SOLVING CLOSED JACKSON QUEUEING NETWORK PROBLEM

Aman Kumar Sinha[1], Dr. S.P. Varma[2]

[1]*Research Scholar in I.T., B. R. A.Bihar University, Muzaffarpur. Email- sinha.aman8624@gmail.com*

[2]*Ex- Associate Professor, P.G Department of Mathematics,*

B. R. A. Bihar University, Muzaffarpur.

Abstract

In the study of queueing networks, closed queueing networks had a major role in the performance analysis of computer systems, communication networks and other related systems. Computer communications and manufacturing system are major areas where queueing theory plays an important role. In this article, we propose to concentrate on closed Jackson network.

In what follows, we proceed to study closed Jackson network that satisfies normalizing condition. The normalizing condition that the sum of all possible states (k_1, $k_{2 ....,}k_n$) that satisfy the condition $\sum_{j=1}^{N} K$ with ($0\leq k_j \geq K$) must be 1.

Keywords-Queueing Networks, closed Jackson network, normalizing condition.

Introduction:

A queueing network is called a closed Jackson network if new customers never enter the system and existing customers never leave the system. That is in this case arrival rate λ_i= 0 and P_{i0}= 0 for all *i*. Thus it becomes a finite source queueing system of N customers who traverse inside the network continuously and service of server *i* is exponentially distributed with rate μ_i, where $i = 1,2, \dots \dots, k$.

Suppose λ_i=0 and P_{i0} = 0 in the assumptions made while defining the open Jackson network. Let $Q = \sum_1^k Q_i$ be the total number of customers in the network. Now we have a closed Jackson network, which can be used to model a network of queues with a fixed number of customers moving within the system.

As in the open queueing networks with *k* nodes and *i*th node supporting s_i servers where *i* =1,2,………..*k*.

$$p_{n_1 n_2 \dots\dots\dots\dots n_k} = C\Pi_{i=1}^{k} \frac{{\rho_i}^{n_i}}{a_i(n_i)}, \dots\dots\dots \text{(i)}$$

where

$$a_i(n_i) = \begin{cases} n_i!, & n_i < s_i \,, \\ s_i!\, {s_i}^{n_i - s_i} & n_i \geq s_i \,, \end{cases}$$

and $\rho_i = \frac{\gamma_i}{\mu_i}$ with γ_i satisfying the relation

$$\gamma_i = \sum_{j=1}^{k} \gamma_j P_{ji.}$$

This relation can be written as

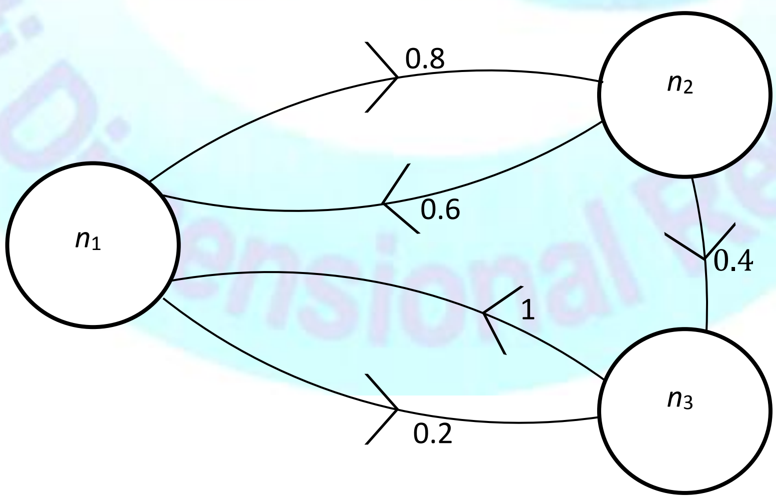
$$\mu_i \rho_i = \sum_{j=1}^{k} \mu_j \rho_j P_{ji.}$$

The constant term C in equation (i) can be determined using the normalizing condition $\sum_{n_1 n_2 \ldots\ldots\ldots\ldots n_k} p_{n_1 n_2 \ldots\ldots\ldots\ldots n_k} = 1$

We are giving below an illustrative example for solution of a closed Jackson network problem that obeys Poisson process.

In a factory there are 2 machines which are expected to be operational at all times and 2 service men, one of whom will rectify ordinary defects and the other will do the service in respect of serious defects. The machines break down according to an exponential distribution with parameter 2. When a machine breaks down, it has a probability 0.8 of being serviced for ordinary defect and a probability 0.2 of being serviced for serious defect. After service for ordinary defect, a machine will require service for serious defect with probability 0.4 and return to operation otherwise. After service for serious defect, the machine will always become operational. Treating the operational status of the machines as node 1, find the probability that –

(i) both the servicemen are idle
(ii) at least one machine is operational. Assume that the service times of the repairmen are exponentially distributed with parameters 3 and 4 respectively.



The situation in given above is a Jackson's closed network.

We assume that the processing point by the machines is node 1 and the service points by two service men are node 2 and node 3.

Now μ_1=2, μ_2=3, μ_3=4 and P_{12} = 0.8, P_{13} = 0.2, P_{21} = 0.6, P_{23} = 0.4, P_{31} = 1

P (n_1, n_2, n_3) means the probability that at n_1 node there are two machines, at n_2 node there is one service man who removes the ordinary defect and at n_3 node there is an another service man who shorts out serious defects.

There are 6 possible combinations for the values of n_1, n_2, n_3 namely (2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1) and (0,1,1), since $n_1 + n_2 + n_3$=2.

The flow balance equations $\gamma_i = \sum_{j=1}^{3} \gamma_j P_{ji}$ (i=1, 2, 3)

Put in the matrix form as

$$(\gamma_1, \gamma_2, \gamma_3) = (\gamma_1, \gamma_2, \gamma_3) \begin{pmatrix} 0 & P_{12} & P_{13} \\ P_{21} & 0 & P_{23} \\ P_{31} & P_{32} & 0 \end{pmatrix} \quad \text{......................} (1)$$

Putting $\rho_i = \frac{\gamma_i}{\mu_i}$ in equation (1)

$$= \quad (\rho_1\mu_1, \rho_2\mu_2, \rho_3\mu_3) \begin{pmatrix} 0 & P_{12} & P_{13} \\ P_{21} & 0 & P_{23} \\ P_{31} & P_{32} & 0 \end{pmatrix}$$

$$= \quad (\rho_1\mu_1, \rho_2\mu_2, \rho_3\mu_3) \begin{pmatrix} 0 & 0.8 & 0.2 \\ 0.6 & 0 & 0.4 \\ 1 & 0 & 0 \end{pmatrix}$$

i.e., $\rho_2\mu_2$ x 0.6 + $\rho_3\mu_3$ x 1 = 2 ρ_1

⇒1.8 ρ_2 + 4 ρ_3 = 2 ρ_1.................... (2)

i.e., 0.8$\rho_1\mu_1$= 0.8 ρ_1x 2 =$\rho_2\mu_2$

⇒ 3ρ_2=1.6 ρ_1

∴ $\rho_1 = \frac{15}{8}\rho_2$....................................... (3)

i.e., 0.2$\rho_1\mu_1$ + 0.4 x $\rho_2\mu_2$= $\rho_3\mu_3$

⇒4ρ_3 = 0.2 x 2 ρ_1+0.4 x 3 x ρ_2

$\Rightarrow 4\rho_3 = 0.4\,\rho_1 + 1.2\,\rho_2$

$\Rightarrow 10\rho_3 = \rho_1 + 3\,\rho_2$……………………. (4)

Equations (2), (3) and (4) are homogeneous linear equations in ρ_1, ρ_2 and ρ_3. To get one solution, we put ρ_2=1 and solve the equations.

Now putting the value of ρ_2=1 in equation (3)

We get, $\rho_1 = \frac{15}{8}$

Now putting the value $\rho_1 = \frac{15}{8}$ in equation (2)

We get, $\rho_3 = \frac{39}{80}$

Since there are2 machines at node 1, the system is a 2-server queueing model.

$\therefore P\,(n_1, n_2, n_3) = C_N \frac{\rho_1^{n_1}}{a_1(n_1)} . \rho_2^{n_2} . \rho_3^{n_3}$,

Where $N = n_1 + n_2 + n_3 = 2$

Now $C_N^{-1} = \sum_{n_1 + n_2 + n_3 = 2} \frac{\rho_1^{n_1}}{n_1!} . \rho_2^{n_2} . \rho_3^{n_3}$, since

$$a_1\,(n_1) = \begin{cases} n_1! & \text{when } n_1 < s_1 \\ s_1! . {s_1}^{n_1 - s_1} & \text{when } n_1 \geq s_1 \end{cases}$$

for all possible combinations of n_1, n_2, n_3 such that $n_1 + n_2 + n_3 = 2$

Thus $C_N^{-1} = \frac{\rho_1^2}{2!}\rho_2^0 . \rho_3^0 + \frac{\rho_1^0}{0!}\rho_2^2 . \rho_3^0 + \frac{\rho_1^0}{0!}\rho_2^0 . \rho_3^2 + \frac{\rho_1^1}{1!}\rho_2^1 . \rho_3^0 + \frac{\rho_1^1}{1!}\rho_2^0 . \rho_3^1 + \frac{\rho_1^0}{0!}\rho_2^1 . \rho_3^1$

$= \left(\frac{15}{8}\right)^2 \times \frac{1}{2} + 1 + \left(\frac{39}{80}\right)^2 + \frac{15}{8} + \frac{15}{8} \times \frac{39}{80} + \frac{39}{80}$

$= 1.7578 + 1 + 0.2376 + 1.875 + 0.9140 + 0.4875$

$= 6.2719$

$C_N = 0.1594$

Now, $P(2,0,0) = 0.1594 \times \left(\frac{15}{8}\right)^2 \times \frac{1}{2} = 0.2802$; $P(0,2,0) = 0.1594$

$P(0,0,2) = 0.0378$; $P(1,1,0) = 0.2988$; $P(1,0,1) = 0.1456$

$P(0,1,1) = 0.0777$

P(both the service man is idle) = *P*(2,0,0) = 0.2802

P(at least one machine is operational) = *P*(1,1,0) + *P*(1,0,1) + *P*(2,0,0)

= 0.2988 + 0.1456 + 0.2802

= 0.7246

CONCLUSIONS

We have solved queueing network problem that obeys Poisson process and calculated probability of both service men remaining idle when two machines work efficiently as well as when one machine is operational.

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