Integral solution of the non-homogeneous Quintic Equation with Seven Unknowns

$$xy(x^2+y^2)-zw(z^2+w^2)=4\sigma^2 XYT^3.$$

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ABSTRACT

We obtain infinitely many non-zero integer solutions (x,y,z,w,X,Y,T) satisfying the non- homogeneous quintic equation with seven unknowns given by $xy(x^2+y^2)-zw(z^2+w^2)=4\sigma^2 XYT^3$ Various interesting relations between the solutions and special numbers, namely, polygonal numbers, Pyramidal numbers, Stella Octangular numbers, Octahedral numbers,, Jacobsthal number, Jacobsthal-Lucas number, keynea number, Centered pyramidal numbers are presented

Keywords:

Centered pyramidal numbers, Integral solutions, Non-homogeneous equation, Polygonal numbers, Pyramidal numbers.

MSC 2000 Mathematics subject classification: 11D41.

Notations:

$T_{m,n}$ -Polygonal number of rank n with size m

P_n^m - Pyramidal number of rank n with size m

S_n - Star number of rank n

J_n -Jacobsthal number of rank of n

j_n - Jacobsthal-Lucas number of rank n

KY_n -keynea number of rank n

$CP_{n,6}$ - Centered hexagonal pyramidal number of rank n

I. INTRODUCTION

The theory of diophantine equations offers a rich variety of fascinating problems.
In particular, biquadratic Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity (**Dickson L.E** (1952), [1], **MordellL.J** (1969), [2], **Carmichael R.D** (1959), [3])
For illustration,one may refer **Gopalan M.A et al** (2010), [4], [5],(2011), [7], (2013),[6], [8] **Vidhyalakshmi et al** (2013), [9], [10], [11], [12] for homogeneous and non-homogeneous quintic equations with three, four and five unknowns.

This paper concerns with the problem of determining non-trivial integral solution of the non-homogeneous quintic equation with seven unknowns given by $xy(x^2+y^2)-zw(z^2+w^2)=4\sigma^2 XYT^3$.A few relations between the solutions and the special numbers are presented.

II. METHOD OF ANALYSIS

The Diophantine equation representing the non- homogeneous quintic equation is given by

$$xy(x^2+y^2)-zw(z^2+w^2)=4\sigma^2 XYT^3 \qquad (1)$$

Introduction of the transformations

$$x=\sigma u+p, y=\sigma u-p, z=\sigma v+p, w=\sigma v-p, X=\sigma(u+v), Y=\sigma(u-v) \qquad (2)$$

in (1) leads to

$$u^2+v^2=2T^3 \qquad (3)$$

The above equation (3) is solved through different approaches and thus, one obtains different sets of solutions to (1)

A. Pattern1:

2 can be written as

$$2=(1+i)(1-i) \qquad (4)$$

Let $T=a^2+b^2$ (5)

Substituting (5) and (4) in (3) and using the method of factorisation, define,

$$(u+iv)=(1+i)(a+ib)^3 \qquad (6)$$

Equating real and imaginary parts in (6) we get

$$\left.\begin{aligned} u&=(a^3-3ab^2)-(3a^2b-b^3)\\ v&=(3a^2b-b^3)+(a^3-3ab^2)\end{aligned}\right\} \qquad (7)$$

In view of (2), (5) and (7), the corresponding values of x, y, z, w, X, Y, T are represented by

$$\left.\begin{aligned} x&=\sigma(a^3-3ab^2-3a^2b+b^3)+p\\ y&=\sigma(a^3-3ab^2-3a^2b+b^3)-p\\ z&=\sigma(3a^2b-b^3+a^3-3ab^2)+p\\ w&=\sigma(3a^2b-b^3+a^3-3ab^2)-p\\ X&=2\sigma(a^3-3ab^2)\\ Y&=2\sigma(3a^2b-b^3)\\ T&=a^2+b^2\end{aligned}\right\} \qquad (8)$$

The above values of x, y, z, w, X, Y and T **satisfies the following relations:**

1. $x(a,a)+y(a,a)+z(a,a)+w(a,a)+8\sigma CP_{a,6}=0$
2. The following expressions are nasty numbers.

(a) $3\sigma[Y(a,1)+6\sigma T_{4,a}]$

(b) $3p[z(a,a)-w(a,a)+T(a,a)-2T_{4,a}]$

(c) $6p[x(a,b)-y(a,b)+z(a,b)-w(a,b)]$

3.The following expressions are cubical integers:

(a) $4\sigma^2[X(a,1)+6\sigma(2T_{3,a}-T_{4,a})]$

(b) $2p^2[x(a,b)-y(a,b)+z(a,b)-w(a,b)]$

4. $X(a,1)+Y(a,1)-\sigma(2CP_{a,6}-S_a+3)\equiv 0(\mathrm{mod}12)$

5. $T(2^{2n},2^{2n})=2(KY_{2n}-j_{2n+1})$

6. $4p^3[x(a,a)-y(a,a)+z(a,a)-w(a,a)+X(a,a)-Y(a,a)]$ is a biquadratic integer.

7. $X(a,a)+Y(a,a)+x(a,a)+y(a,a)+16\sigma(2P_a^5-T_{4,a})=0$

8. $x(a,1)+z(a,1)-X(a,1)+T(a,1)-2p-T_{4,a}=1$

9. $x(a,b)+y(a,b)+z(a,b)+w(a,b)=2X(a,b)$

10. $x(a,b)+y(a,b)-X(a,b)-Y(a,b)=0$

11. $x(a,b).y(a,b)-z(a,b).w(a,b)=X(a,b).Y(a,b)$

12. $\sigma(4P_a^5-2CP_{a,6}+2T_{4,b})-X(a,b)-2\sigma T(a,b)\equiv 0(\mathrm{mod}\,6)$

B. Pattern2:

2 can also be written as

$$2=\frac{(7+i)(7-i)}{5^2} \tag{9}$$

Substituting (5) and (9) in (3) and using the method of factorisation, define,

$$(u+iv)=\frac{(7+i)}{5}(a+ib)^3 \tag{10}$$

Using the same procedure as in Pattern1 the integral solution of (1)

$$\left.\begin{aligned}
x &= 25\sigma[7(A^3-3AB^2)-3A^2B+B^3]+p\\
y &= 25\sigma[7(A^3-3AB^2)-3A^2B+B^3]-p\\
z &= 25\sigma[A^3-3AB^2+7(3A^2B-B^3)]+p\\
w &= 25\sigma[A^3-3AB^2+7(3A^2B-B^3)]-p\\
X &= 25\sigma[8(A^3-3AB^2)+6(3A^2B-B^3)]\\
Y &= 25\sigma[6(A^3-3AB^2)-8(3A^2B-B^3)]\\
T &= 25(A^2+B^2)
\end{aligned}\right\} \tag{11}$$

Remark: 1

2 can also be written as

$$2=\frac{(41+i)(41-i)}{29^2}$$

Using the same procedure as in Pattern1 the integral solution of (1) can be obtained.

C. Pattern3:

Substitution of $u = P + Q, v = P - Q$ (12)

in (3) reduces it to

$$P^2 + Q^2 = T^3 \quad (13)$$

The solution to (13) is obtained as

$$P = \alpha(\alpha^2 + \beta^2), T = \alpha^2 + \beta^2, Q = \beta(\alpha^2 + \beta^2) \quad (14)$$

In view of (14), (12) and (2) the integral solution of (1) is obtained as

$$\left.\begin{array}{l} x = \sigma(\alpha+\beta)(\alpha^2+\beta^2) + p \\ y = \sigma(\alpha+\beta)(\alpha^2+\beta^2) - p \\ z = \sigma(\alpha-\beta)(\alpha^2+\beta^2) + p \\ w = \sigma(\alpha-\beta)(\alpha^2+\beta^2) - p \\ X = 2\sigma\alpha(\alpha^2+\beta^2) \\ Y = 2\sigma\beta(\alpha^2+\beta^2) \\ T = \alpha^2+\beta^2 \end{array}\right\} \quad (15)$$

Remark2:

The solution to (13) can also be obtained as

$P = \alpha^3 - 3\alpha\beta^2, Q = 3\alpha^2\beta - \beta^3, T = \alpha^2 + \beta^2$ Substituting the above result in (12) and using (2), the corresponding integral solution of (1) can be obtained.

D. Pattern4:

The assumption

$$P = P'T, Q = Q'T \quad (16)$$

in (13) yields to

$$P'^2 + Q'^2 = T \quad (17)$$

(i) Taking $T = t^2$ (18)

in (17), we get

$$P'^2 + Q'^2 = t^2 \quad (19)$$

Then the solutions to (19) is given by

$$Q' = 2\alpha\beta, t = \alpha^2 + \beta^2, P' = \alpha^2 - \beta^2, \;\; \alpha > \beta > 0 \quad (20)$$

$$P' = 2\alpha\beta, t = \alpha^2 + \beta^2, Q' = \alpha^2 - \beta^2, \;\; \alpha > \beta > 0 \quad (21)$$

From (20), (18) (16) and (12) we get

$$\left.\begin{array}{l} u = (\alpha^2+\beta^2)^2(\alpha^2-\beta^2+2\alpha\beta) \\ v = (\alpha^2+\beta^2)^2(\alpha^2-\beta^2-2\alpha\beta) \\ T = (\alpha^2+\beta^2)^2 \end{array}\right\} \quad (22)$$

In view of (22) and (2), we get the corresponding integral solution of (1).as

$$\left.\begin{aligned}
x &= \sigma(\alpha^2+\beta^2)^2(\alpha^2-\beta^2+2\alpha\beta)+p\\
y &= \sigma(\alpha^2+\beta^2)^2(\alpha^2-\beta^2+2\alpha\beta)-p\\
z &= \sigma(\alpha^2+\beta^2)^2(\alpha^2-\beta^2-2\alpha\beta)+p\\
w &= \sigma(\alpha^2+\beta^2)^2(\alpha^2-\beta^2-2\alpha\beta)-p\\
X &= 2\sigma(\alpha^2+\beta^2)^2(\alpha^2-\beta^2)\\
Y &= 4\alpha\beta\sigma(\alpha^2+\beta^2)^2\\
T &= (\alpha^2+\beta^2)^2
\end{aligned}\right\} \quad (23)$$

Remark3:
Similarly by considering (21), (18), (16), (12) and (2), we get the corresponding integral solution to (1).

(ii) Now, rewrite (13) as,

$$P^2+Q^2=1*T^3 \quad (24)$$

Also 1 can be written as

$$1=(-i)^n(i)^n \quad (25)$$

Let $T=a^2+b^2$ (26)

Substituting (25) and (26) in (23) and using the method of factorisation, define,

$$(P+iQ)=i^n(a+ib)^3 \quad (27)$$

Equating real and imaginary parts in (27) we get

$$\left.\begin{aligned}
P &= \cos\frac{n\pi}{2}(a^3-3ab^2)-(3a^2b-b^3)\sin\frac{n\pi}{2}\\
Q &= (3a^2b-b^3)\cos\frac{n\pi}{2}+\sin\frac{n\pi}{2}(a^3-3ab^2)
\end{aligned}\right\} \quad (28)$$

In view of (28), (11) and (2) we get the integral solution of (1)

$$\left.\begin{aligned}
x &= \sigma f(a,b)+p\\
y &= \sigma f(a,b)-p\\
z &= \sigma g(a,b)+p\\
w &= \sigma g(a,b)-p\\
X &= \sigma[f(a,b)+g(a,b)]\\
Y &= \sigma[f(a,b)-g(a,b)]\\
T &= a^2+b^2
\end{aligned}\right\} \quad (29)$$

where

$$f(a,b)=[\cos\frac{n\pi}{2}+\sin\frac{n\pi}{2}](a^3-3ab^2)+[\cos\frac{n\pi}{2}-\sin\frac{n\pi}{2}](3a^2b-b^3)$$

$$g(a,b)=[\cos\frac{n\pi}{2}-\sin\frac{n\pi}{2}](a^3-3ab^2)-[\cos\frac{n\pi}{2}+\sin\frac{n\pi}{2}](3a^2b-b^3)$$

(iii) Taking $T=t^n$ and $t=a^2+b^2$ (30)
in (17), we get

$$P'^2 + Q'^2 = t^n$$

Let

$$t^n = (a+ib)^n (a-ib)^n$$

$$P' + iQ' = (a+ib)^n$$

$$P' - iQ' = (a-ib)^n$$

$$\left.\begin{aligned} P' &= \frac{1}{2}[(a+ib)^n + (a-ib)^n] \\ Q' &= \frac{1}{2i}[(a+ib)^n - (a-ib)^n] \end{aligned}\right\} \quad (31)$$

In view of (2), (12), (16), (30) and (31), the corresponding values of x, y, z, w, X, Y, T are represented as

$$x = \left(\frac{1}{2} f + \frac{1}{2i} g\right)(a^2+b^2)^n \sigma + p$$

$$y = \left(\frac{1}{2} f + \frac{1}{2i} g\right)(a^2+b^2)^n \sigma - p$$

$$z = \left(\frac{1}{2} f - \frac{1}{2i} g\right)(a^2+b^2)^n \sigma + p$$

$$w = \left(\frac{1}{2} f - \frac{1}{2i} g\right)(a^2+b^2)^n \sigma - p$$

$$X = \sigma(a^2+b^2)^n f$$

$$Y = -\sigma i(a^2+b^2)^n g$$

$$T = (a^2+b^2)^n$$

where

$$\left.\begin{aligned} f &= [(a+ib)^n + (a-ib)^n] \\ g &= [(a+ib)^n - (a-ib)^n] \end{aligned}\right\} \quad (32)$$

(iv)1 can also be written as

$$1 = \frac{((m^2-n^2)+i2mn)((m^2-n^2)-i2mn)}{(m^2+n^2)^2} \quad \text{(or)}$$

$$1 = \frac{(2mn+i(m^2-n^2))(2mn-i(m^2-n^2))}{(m^2+n^2)^2}$$

and performing the same procedure as above the corresponding integral solution to (1) can be obtained

III. CONCLUSION

In conclusion, one may search for different patterns of solutions to (1) and their **c**orresponding properties.

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