POLYNOMIAL IDENTITY RINGS AND GROUPRINGS

PREETI PANWAR ,Assistant Professor In Mathematics

GURU NANAK KHALSA COLLEGE ,KARNAL

Abstract: In this paper,we prove the sufficient condition for group rings to satisfy a polynomial identity. Firstly we define groupring and standard polynomial identity in polynomial ring $K[X_1, X_2, \ldots]$ over field K with noncommuting indeterminates $X_1, X_2, \ldots$.

Introduction: *Let K be a field and G be a multiplicative* group. Let K[G] denote the set of all formal sums k=$\sum_{g\in G} k_g\, g$, where $k_g \in K$ for every g$\in G$ and the set $\{g \in G/k_g \neq 0\}$ is finite. For k=$\sum_{g\in G} k_g\, g$ and s=$\sum_{g\in G} s_g\, g$ belonging to K[G],define k+s= $\sum_{g\in G}(k_g + s_g)$. This defines addition in K[G] with respect to which K[G] becomes an abelian group. Again for $\alpha \in K$ and k=$\sum_{g\in G} k_g\, g \in K[G]$, we define $\alpha k = \sum_{g\in G}(\alpha k_g)g$. With respect to this scalar multiplication, K[G] is vectorspace over K. For k=$\sum_{g\in G} k_g\, g$ and s=$\sum_{g\in G} s_h\, h$,define $ks = \sum_{l\in G} t_l\, l$ where $t_l = \sum_{gh=l} k_g s_h$ and the elements of G commutes with the elements of K. With the multiplication as defined above, K[G]becomes a ring. Hence K[G] is an algebra over the field K and is called group ring over K.

Next we define polynomial identity. Let $K[X_1, X_2, \ldots]$ be the polynomial ring over a field K in the noncommuting indeterminates $X_1, X_2, \ldots$. An algebra E over K is said to satisfy a polynomial identity,if there exists $f(X_1, X_2, \ldots, X_n) \in K[X_1, X_2, \ldots], f \neq 0\ \forall \alpha_1, \alpha_2, \ldots, \alpha_n \in E$.

For example,any commutative algebra satisfies $f(X_1, X_2) = X_1X_2 - X_2X_1$.

The standard polynomial of degree n is defined by

$[X_1, X_2, \ldots, X_n]$=$\sum_{\sigma\in S_n}(-1)^{\sigma}\, X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)}$.

Here S_n is the symmetric group of degree n on the set $S = \{1,2, \ldots, n\}$ and $(-1)^{\sigma}$ is 1 or -1 according as σ is an even or an odd permutation. We will also use $s_n(X_1, X_2, \ldots, X_n)$ to denote this polynomial.

Theorem 1. Let the group G have an abelian subgroup A such that $[G: A] = n < \infty$ .Then $K[G]$ satisfies the standard polynomial identity of degree 2m.

Proof . Let $x_1, x_{2,\ldots,} x_n$ be a set of right coset representatives for A in G. Let E=K[A] andV=K[G]. Since A is abelian therefore Eis commutative algebra. Clearly Vis left E-module. Since G is a basis of K[G] over the field K and $G = \bigcup_{i=1}^{n} Ax_i$ therefore $\{x_1, x_2, \dots, x_n\}$ is a basis of V over E. Now V is also a right K[G]-module and as such it is faithful.

Let $E_n = M_n(E)$ be the ring of all $n \times n$ matrices over E. We define : $K[G] \rightarrow Hom_{K[A]}(K[G], K[G]) = R\ by\ \phi(u) = T_u\ where\ T_u: K[G] \rightarrow K[G]$ is defined by $v\mathrm{T}_u = vu\ \forall\ v \in K[G]$. Let $\phi(u) = 0$ for some $u \in K[G]$, which implies $T_u = 0$ and $\therefore vu = 0\ \forall\ v \in K[G]$. Hence $u \in Ann(K[G]) = Ann(V) = 0$. Therefore ϕ is one-one. Hence $V = K[G] \cong \phi(V) \subseteq Hom_E(V, V) = R$. Define $\phi: R \rightarrow M_n(E) = E_n$ as follows.

Let $T \in R$ then $x_i T \in V$. Since $\{x_1, x_2, \dots, x_n\}$ is a basis of V over E therefore there exist $a_{i1}, a_{i2}, \dots, a_{in} \in E$ such that $x_i T = \sum_{j=1}^{n} a_{ij\, x_j}$. We define $\phi(T) = m(T)\ where\ m(T) = \left[a_{ij}\right]_{n \times n} \in E_n$. It is easy to see that ϕ is an algebra isomorphism and hence $\subseteq R \cong E_n$. Now define a map $f: E \times K_n \rightarrow E_n\ by\ f(a, A) = aA\ where\ K_n = M_n(K)$. Let $F(E, M_n(K)$ be a free Z-module on $E \times K_n$ and $G(E, K_n)$ be the submodule of $F(E, K_n)$ generated by the elements of the form

i) $(a_1 + a_2, A) - (a_1, A) - (a_2, A)$
ii) $(a, A + B) - (a, A) - (a, B)$
iii) $(ab, A) - (a, bA)$ where $a_1, a_2, a_n, a, b \in$ E; $A, B \in K_{n.}$

Extend the map to a Z-homomorphism f of $F(\mathrm{E}, \mathrm{K}_n)$ into E_n which clearly vanishes on $G(E, K_n)$ and therefore induces a Z- homomorphism $f: E \otimes K \rightarrow E_n$ given by $(\sum a_i \otimes A_i = \sum a_i A_i$. Define $g: E_n \rightarrow E \otimes K_n$ by $g(A) = g\left(\sum_{i,j} a_{ij}\, e_{ij}\right) = \sum_{i,j} a_{ij} \otimes e_{ij}$ where $A = [a_{ij}] \in E_n$ and $\{e_{ij}\}$ are the matrix units in E_n then it is easy to see that g is a Z-homomorphism. Now $(fog)(A) = f\big(g(A)\big) = f\left(\sum_{i,j} a_{ij} \otimes e_{ij}\right) = \sum_{i,j} a_{ij}\, e_{ij} = A$ where $A = [a_{ij}] \in E_n$. Hence $fog = I_{E_n}$. Also it$\left(a_{\sigma(1)} \otimes A_{\sigma(1)}\right)$ is easy to see that $gof = I_{E \otimes K_n.}$ Hence $\otimes K_n \cong E_n$. Thus $K[G] \subseteq E \otimes K_{n.}$ By the result "K_m, the ring of all $m \times m$ matrices over a field K, satisfies the standard polynomial identity of degree $2m$", K_n satisfies s_{2n}.

We prove that $E \otimes K_n$ also satisfies s_{2n}. Let $a_1 \otimes A_1, a_2 \otimes A_2, \dots a_{2n} \otimes A_{2n} \in E \otimes K_n$ then

$$s_{2n(a_1 \otimes A_1, a_2 \otimes A_2, \dots a_{2n} \otimes A_{2n})} = \sum_{\sigma \epsilon\, S_{2n}} (-1)^{\sigma} \left(a_{\sigma(1)} \otimes A_{\sigma(1)}\right)\left(a_{\sigma(2)} \otimes A_{\sigma(2)}\right) \dots \left(a_{\sigma(2n)} \otimes A_{\sigma(2n)}\right)$$

where S_{2n} is symmetric group of degree 2n on set $\{1,2, \dots, 2n\}$.

$$= \sum_{\sigma \in S_{2n}} (-1)^{\sigma} \left(a_{\sigma(1)} a_{\sigma(2)} \dots a_{\sigma(2n)} \otimes A_{\sigma(1)} A_{\sigma(2)} \dots A_{\sigma(2n)}\right)$$

Since E is commutative ring therefore $a = a_1 a_2 \dots a_{2n} = a_{\sigma(1)} a_{\sigma(2)} \dots a_{\sigma(2n)} \forall \sigma \in S_{2n}$.

$$s_{2n(a_1 \otimes A_1, a_2 \otimes A_2, \dots a_{2n} \otimes A_{2n})} = \sum_{\sigma \in S_{2n}} (-1)^{\sigma} \left(a \otimes A_{\sigma(1)} A_{\sigma(2)} \dots A_{\sigma(2n)}\right)$$

$$= a \otimes \textstyle\sum_{\sigma \in S_{2n}} (-1)^{\sigma} A_{\sigma(1)} A_{\sigma(2)} \dots A_{\sigma(2n)}$$

$$= a \otimes s_{2n}(A_1, A_2, \dots A_{2n})$$

Since K_n satisfies s_{2n} therefore $s_{2n(A_1, A_2, \dots, A_{2n})} = 0$ and hence $s_{2n(a_1 \otimes A_1, a_2 \otimes A_2, \dots a_{2n} \otimes A_{2n})=0}$.

Thus $E \otimes K_n$ satisfies s_{2n} and the result follows.

REFERENCES:

1. Connel,I.G. On the group rings.
2. Herstein,I.N. Non Commutative rings.
3. Lam,T.Y. Non Commutative rings.
4. Passman,D.S. Infinite Group rings.
5. Passman,D.S. The Algebraic Structure of Group Rings.
6. Macdonald, I.D. Theory of Groups.