Torseforming Curvature and Ricci Tensor in a Trans- Sasakian Manifold

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Abstract. The aim of the present paper, I have studied Torseforming Curvature and Ricci Tensor in a Trans-Sasakian Manifold. The sectional curvature of a plane section of such a manifold containing U is a constant, saysC . He divided these manifolds into three cases: (1) homogeneous normal contact Riemannian manifold with C >0, (2) global Riemannian products of a line or a circle with a manifold of constants holomorphic sectional curvature if C = 0 and (3) a warped product space $R_{Xf}\, C^{\infty}$ if C>0.

I know that the manifolds belonging to class (1) is characterized by admitting a Sasakian structure. I have also obtained necessary and sufficient condition that trans-Sasakian manifold is flat. Here we proved that the trans-Sasakian manifold satisfying the condition R(X ,Y)S = 0and torseforming trans-Sasakian manifolds in a curvature and Ricci tensor under these condition $\overline{\varphi\, \text{grad}\alpha} = \text{grad}\beta$, is a consircular tensor.

Key Words: trans-Sasakian manifold, torseforming curvature and Ricci tensor, con-circular tensor.

Introduction. The purpose of the present paper is to define and study the torseforming Curvature and Ricci tensor in a trans-Sasakian manifold. In section 1 we review and collect some necessary results. In section 2 I define trans-Sasakian manifolds satisfying the condition R(X ,Y) = 0 . In section 3 I have also define torseforming trans-Sasakian manifolds. The contact manifolds are n = 2m +1 dimensional manifolds with specified contact structure. I can obtain different structure like Sasakian, Quasi Sasakian, Kenmotsu and trans-Sasakian by providing additional conditions. The geometry of the manifolds is extensively studied trans-Sasakian manifold and invariant sub-manifolds of a conformal K-contact Riemannian manifold by [3] to [2]. Now the torseforming vector field in a Riemannian manifold has been introduced by K. Yano in 1944 [10].

In this paper I have also defined a conformally contact Riemannian manifold and established some of its properties and another meaningful results.

1. Preliminaries. Let us consider an n = 2m +1dimension real differentiable manifold with an almost contact metric structure (F,U,u, g) on which there are defined a tensor field of type (1,1), a vector field U and 1-form u satisfying for every vector field X

(1.1) (a) $\overline{\overline{X}} = I + u \quad U,$ (b) $F(X) = \overline{X}$

(c) $u(U) = -\,,1$ (d) $u(\overline{X}) = 0$ (e) $\overline{U} = 0$

and rank$\varphi = n = 2m + 1$.

Where I is the identity endomorphism of the tangent bundle of M^n

(1.2) (a) $g(\overline{X},\overline{Y}) = -g(X,Y) - u(X)u(Y),$

(b) $g(\overline{X}, Y) = -g(X, \overline{Y})$,

(c) $g(X, U) = u(X)$

for all X,Y TM .

An almost contact metric structure (F,U,u, g) on M^n is called a trans-Sasakian structure, then consider the manifold ($M^n \cdot R$, J, g_1) and denoted by $(X, f\frac{d}{dt})$ a vector field of $M^n \cdot R$, where X is a tangent to M^n, t is the field of R and f is a differentiable function on $M^n \cdot R$. An almost complex structure J on this manifold is defined as

(1.3) $$J\left(X, f\frac{d}{dt}\right) = \left(\overline{X} - fU, u(X)\frac{d}{dt}\right)$$

for all vector field X on M^n and smooth function a on $M^n \cdot R$ and g_1 is the product metric on $M^n \cdot R$. It is known that ($M^n \cdot R$, J, g_1) is an almost Hermitian manifold [11] ,where g_1 denote the product metric given by

(1.4) $$g_1\left(\left(X, f\frac{d}{dt}\right), \left(Y, h\frac{d}{dt}\right)\right) = g(X, Y) + fh$$

I now recall the following important result due to Lotta by [1].

Let M^n be a slant sub-manifold of an almost contact metric manifold M^n with slant angle $\theta \neq \frac{\pi}{2}$, then we have

n = 2m +1 is odd U is tangent to M .

This is may be expressed by the condition [7]

(1.5) $$\nabla_X \overline{Y} = \alpha\{g(X, Y)U - u(Y)X\} + \beta\{g(\overline{X}, Y)U - u(Y)\overline{X}\},$$

for some smooth functions α and β on M^n and we say that trans-Sasakian structure is of type (α, β) .From (1.5) it follows that

(1.6) (a) $$\nabla_X U = -\alpha(\overline{X}) + \beta\{X - u(X)U\}$$

(b) $$(\nabla_X u)Y = -\alpha g(\overline{X}, Y) + \beta g(\overline{X}, \overline{Y})$$

I note that the trans-Sasakian structures of type (0,0) are cosymplectic trans-Sasakian structures of type $(0, \beta)$ are $\beta-$ Kenmotsu and trans-Sasakian structures of type $(\alpha)0$ are $\alpha-$ Sasakian. Thus, in trans-Sasakian structures of type (0,0), the equations (1.5) and (1.6) reduce to

(1.7) $$\nabla_X \overline{\ } = 0, \qquad \nabla_X U = 0.$$

For $\beta-$ Kenmotsu structure (1.5) and (1.6) reduce to

(1.8) (a) $$\nabla_X \overline{Y} = \beta\{g(\overline{X}, Y)U - u(Y)\overline{X}\}$$

(b) $$\nabla_X U = \beta\{X - u(X)U\},$$

while for $\alpha-$ Sasakian. Structure, we have

(1.9) (a) $$\nabla_X \overline{Y} = \alpha\{g(\overline{X}, Y)U - u(Y)X\},$$

(b) $_X U = -\alpha(\overline{X})$,

If $\alpha = 1$, $\alpha -$ Sasakian. Structure reduce to Sasakian on trans-Sasakian manifolds, we have the following result.

(1.10) $R(X,Y)U = (\alpha^2 - \beta^2)\{u(Y)X - u(X)Y\} + 2\alpha\beta\{u(Y)\overline{(X)} - u(\overline{X})Y\} + Y\overline{\alpha}(X)$
$- X\alpha(\overline{Y}) + Y\overline{\overline{\beta}}(X) - X\overline{\overline{\beta}}(Y)$,

(1.11) $R(U, X) = (\alpha^2 - \beta^2 - U\beta)\{u(X)U - X\}$,

(1.12) $2\alpha\beta + U\alpha = 0$,

(1.13) $S(X,U) = \{2m(\alpha^2 - \beta^2) - U\beta\}u(X) - (2m-1)X\beta - \overline{(X)}\alpha$,

(1.14) $QU = \{2m(\alpha^2 - \beta^2) - U\beta\}U - (2m-1)\mathrm{grad}\beta + \mathrm{grad}\alpha$,

When $\overline{\varphi\, \mathrm{grad}\alpha} = (2m-1)\mathrm{grad}\beta$, from equations (113), and (1.14) reduce to

(1.15) $S(X,U) - 2m(\alpha^2 - \beta^2)u(X)$,

(1.16) $QU = 2m(\alpha^2 - \beta^2)U$.

Also under the above condition $\overline{\varphi\, \mathrm{grad}}\alpha = (2m-1)\mathrm{grad}\beta$ the expression for Ricci tensor and scalar curvature in a trans-Sasakian manifold are given respectively by

(1.17) $S(X,Y) = \frac{r}{2} - (\alpha^2 - \beta^2)\, g(X,Y) - \frac{r}{2} - 3(\alpha^2 - \beta^2)\, u(X)u(Y)$

and

(1.18) $R(X,Y)Z = \frac{r}{2} - 2(\alpha^2 - \beta^2)\left[g(Y,Z)X - g(X,Z)Y\right] - \frac{r}{2} - 3(\alpha^2 - \beta^2)$
$\left[g(Y,Z)u(X)U - g(X,Z)u(Y)U\right] - \frac{r}{2} - 3(\alpha^2 - \beta^2)$
$\left[u(Y)u(Z)X - u(X)u(Z)Y\right]$,

A vector field ρ defined by $g(X,\rho) = w(X)$ for any vector field X is said to be a torseforming curvature and Ricci tensor

(1.19) $({}_X w)Y = kg(X,Y) + \pi(X)w(Y)$,

where k is a non-zero scalar and π is a non-zero 1-form.

2. Curvature and Ricci tensor in a trans-Sasakian manifold satisfying the condition $R(X,Y) = 0$.

Let us consider a trans-Sasakian manifold M^n which satisfying the condition

(2.1) $R(X,Y) = 0$.

From (2.1),

(2.2) $S(R(X,Y)U,u) + S(U, R(X,Y)u) = 0$.

Putting $X = U$ and using (1.10) and (1.1), we have

(2.3) $R(U,Y)U = (\alpha^2 - \beta^2)\{u(Y)U + Y\} + 2\alpha\beta\overline{Y} - U\alpha(\overline{Y}) - U\overline{\overline{\beta}}(Y)$

using (111) in (2.3), we have

(2.4) $(\alpha^2 - \beta^2)\{u(Y)U + Y - u(X)U + X\} + U\beta\{u(X)U - X + \overline{\overline{Y}}$
$+ 2\alpha\beta\overline{Y} - U\alpha(\overline{Y}) = 0$.

Let $\{e_i\}, i = 1\ 2\ 3............n$ be an orthonormal basis of the tangent space at each point of the manifold M^n. Then putting $X = Y = e_i$ in (2.4) and taking summation for $1 \le i \le n$, we obtain

$$(2.5) \quad 2(\alpha^2 - \beta^2)e_i + e_i\beta\{u(e_i)e_i - e_i + \overline{\overline{e_i}} + 2\alpha\beta\overline{e_i} - \|\alpha e_i\| = 0$$

and using (1.12), we have

$$2\alpha\beta + e_i\alpha = 0$$

$$2\beta + e_i = 0,$$

$$e_i = -\frac{\beta}{2},$$

Substituting the value of e_i in the above equation (2.5), we get

$$(2.6) \quad -\beta\alpha^2 - 2\alpha\beta^2 - \frac{\beta^2}{2}\overline{\overline{Y}} + \frac{3\beta^3}{4} - \frac{\beta^4}{2}u - \alpha\left\|\frac{\beta}{2}\right\| = 0$$

3. Torseforming Curvature and Ricci tensor in a trans-Sasakian manifold

Consider a unit torseforming vector field $\overline{\rho}$ in a tensor form corresponding to a vector field ρ. Suppose $g(X, \overline{\rho}) = T(X)$, then the metric tensor is

$$(3.1) \quad T(X) = \frac{w(X)}{\sqrt{w(\rho)}}.$$

From (1.19) divided into $\frac{1}{\sqrt{w(\rho)}}$ both sides, we get

$$(3.2) \quad \frac{(\nabla_X w)Y}{\sqrt{w(\rho)}} = \frac{k}{\sqrt{w(\rho)}} g(X, Y) + \frac{\pi}{\sqrt{w(\rho)}} w(Y),$$

using equation (3.1) in the equation, we obtain

$$(3.3) \quad (\nabla_X T)Y = \frac{k}{\sqrt{w(\rho)}} g(X, Y) + \pi(X)T(Y),$$

Putting $Y = \overline{\rho}$ in (3.3), we obtain

$$(3.4) \quad (\nabla_X T)\overline{\rho} = \frac{k}{\sqrt{w(\rho)}} g(X, \overline{\rho}) + \pi(X)T(\overline{\rho})$$

As $T(\overline{\rho}) = g(\overline{\rho}, \overline{\rho}) = 1$ is a unit vector then the equation (3.4) reduce to

$$(3.5) \quad (\nabla_X T)\overline{\rho} = \frac{k}{\sqrt{w(\rho)}} g(X, \overline{\rho}) + \pi(X)$$

and hence equation (3.3) can be written as in the form

$$(3.6) \quad (\nabla_X T)(\overline{\rho}) = \frac{k}{\sqrt{w(\rho)}} g(X, Y) + T(X)T(Y).$$

T is closed.

Using covariant differential of (3.6) and using Ricci identity, we get

$$(3.7) \quad -T(R(X, Y)Z) = X\,\kappa\,[g(Y, Z) - T(Y)T(Z)] - Y\,\kappa\,[g(X, Z) - T(X)T(Z)]$$

$$\sqrt{w(\rho)} \qquad \sqrt{w(\rho)}$$

$$+\frac{k^2}{\sqrt{w(\rho)}}\left[g(Y,Z)T(X)-g(X,Z)T(Y)\right].$$

Using equations (1.18) and (3.1) in equation (3.7), we have

(3.8) $$-w\ \frac{r}{2}-2(\alpha^2-\beta^2)\left[g(Y,Z)X-g(X,Z)Y\right]-\frac{r}{2}-3(\alpha^2-\beta^2)$$

$$\left[g(Y,Z)u(X)U-g(X,Z)u(Y)U\right]-\frac{r}{2}-3(\alpha^2-\beta^2)\left[u(Y)u(Z)X-u(X)u(Z)Y\right]$$

$$\frac{r}{2}-3(\alpha^2-\beta^2)\left[u(Y)u(Z)X-u(X)(Z)Y\right]=kX\ g(Y,Z)-\frac{w(Y)w(Z)}{w(\rho)}$$

$$-kY\ g(X,Z)-\frac{w(X)w(Z)}{w(\rho)}+k^2\ g(Y,Z)\frac{w(X)}{\sqrt{w(\rho)}}-g(X,Z)\frac{w(Y)}{\sqrt{w(\rho)}}$$

Putting $Z=\bar{\rho}$ and using $g(X,\bar{\rho})=T(X)$ and equation (3.1), we have

(3.9) $$w\ \frac{r}{2}-2(\alpha^2-\beta^2)\left[g(Y,\bar{\rho})X-g(X,\bar{\rho})Y\right]-\frac{r}{2}-3(\alpha^2-\beta^2)$$

$$\left[g(Y,\bar{\rho})u(X)U-g(X,\bar{\rho})u(Y)U\right]-\frac{r}{2}-3(\alpha^2-\beta^2)\left[u(Y)u(\bar{\rho})X-u(X)u(\bar{\rho})Y\right]$$

$$\frac{r}{2}-3(\alpha^2-\beta^2)\left[u(Y)u(\bar{\rho})X-u(X)(\bar{\rho})Y\right]+kX\ g(Y,\bar{\rho})-\frac{w(Y)w(\bar{\rho})}{w(\rho)}$$

$$-kY\ g(X,\bar{\rho})-\frac{w(X)w(\bar{\rho})}{w(\rho)}+k^2\ g(Y,\bar{\rho})\frac{w(X)}{\sqrt{w(\rho)}}-g(X,\bar{\rho})\frac{w(Y)}{\sqrt{w(\rho)}}=0,$$

(3.10) $$\frac{r}{2}-2(\alpha^2-\beta^2)\left[w(Y)X-w(X)Y\right]\ \frac{r}{2}-3(\alpha^2-\beta^3)\left[w(Y)u(X)U-w(X)u(Y)U\right]$$

$$+\left[1-w(\bar{\rho})\right]\left[kXw(Y)-kYw(X)\right]=0,$$

Putting $X=\bar{\rho}$ in equation (3.10) and $T(\bar{\rho})=g(\bar{\rho},\bar{\rho})=1$, we have

(3.11 $$\frac{r}{2}-2(\alpha^2-\beta^2)\ \left[X-Y\right]-\frac{r}{2}-3(\alpha^2-\beta^2)\left[u(X)U-u(Y)U\right]+(1-w(\bar{\rho}))\left[kX-kY\right]=0$$

Thus, we have the following.

Lemma 1. If a trans-Sasakian manifold admits a torseforming vector field then the following cases occur

(3.12) $u(Y)-u(\bar{\rho})T(Y)=0$,

(3.13) $g(Y,U)\sqrt{w(\rho)}-w(X)w(Y)=0.$

I first consider the case where (3.12) holds well.

From (3.12), we get

$$u(Y)=U(\bar{\rho})T(Y)$$

Now Y = U implies

$$1 = \left(u((\bar{\rho})\right)^2$$

and thus

$$u(\bar{\rho}) = \pm 1.$$

So

(3.14) $u(Y) = \pm T(Y)$,

using (3.14) in (1.6) in view of (3.6), we have

(3.15) $\nabla_X U = \alpha(X) + \beta \quad X - \frac{w(X)}{\sqrt{w(\rho)}} U$,

(3.16) $-\alpha g(\bar{X}, Y) + \beta g(\bar{X}, \bar{Y}) - T(X)T(Y) = \pm \frac{k}{\sqrt{w(\rho)}} \left[g(X, Y) - T(X)T(Y)\right].$

Lemma 2. **The equation (3.12) implies that the vector field $\bar{\rho}$ is a concircular vector field. I next assume the case (3.13), then**

(3.17) $u(Y) - u(\bar{\rho})T(Y) \neq 0$.

From (3.7), we obtain

(3.18) $\quad -T(QX)+ \quad \frac{k}{\sqrt{w(\rho)}} \quad X+\rho \quad - \frac{k}{\sqrt{w(\rho)}} \quad T(X)+ \quad \frac{2k}{w(\rho)}^2 \; T(X)$,

(3.19) $-wQ(X) = kX + \frac{\bar{\rho}}{\sqrt{w(\rho)}} + \frac{2k}{w(\rho)} \quad kw(X)$.

Putting X = U and using (1.16), we have

(3.20) $2m(\alpha^2 - \beta^2) = k + \frac{\bar{\rho}}{\sqrt{w(\rho)}} + \frac{2k}{w(\rho)} \; kw$.

Putting X = U in equation (3.10), in virtue of (3.20) and T(U) = u($\bar{\rho}$) , we get

(3.21) $$\frac{r}{2} - \frac{k}{m} - \frac{kw}{m} \; \frac{\bar{\rho}}{\sqrt{w(\rho)}} + \frac{2k}{w(\rho)} \; [wU(X) - w(X)U]$$

$$-w(X)U \quad \frac{r}{2} - \frac{3k}{2m} - \frac{3kw}{2m} \; \frac{\bar{\rho}}{\sqrt{w(\rho)}} + \frac{2k}{w(\rho)}$$

$$+\left[1 - w(\bar{\rho})\right]\left[kXw(U - kUw(X) = 0\right].$$

From (3.5) it follows that

(3.22) $Y(\nabla_X T)\bar{\rho} = Y \; \frac{k}{\sqrt{w(\rho)}} \; g(X, \bar{\rho}) + \frac{k}{\sqrt{w(\rho)}} \left[Yg(\bar{X}, \bar{\rho})\right] + Y\left[\pi(X)\right],$

since T is closed, π is also closed.

Then we have

Lemma 3. **The above equations imply that the Curvature and Ricci tensor of vector field $\bar{\rho}$ is**

a concircular tensor. Thus from Lemma 2 and Lemma 3, we have state following.
Theorem 3.1 A torseforming tensor in a trans-Sasakian manifold is a concircular tensor field.

From (1.6) it follows that in a trans-Sasakian manifoldU is a torseforming tensor field. Theorem 3.1, I can state the following.
Theorem 3.2 A trans-Sasakian manifold admits a proper concircular tensor field.

Where T is a conformally flat manifold M^n (whose dimension is a grater than n) admits a proper concircular tensor field, then the manifold is a sub-projective manifold. Since the trans-Sasakian manifold M^n admits proper concircular tensor field, namely the tensor fieldU , I can state as follows:

Theorem 3.3 A conformally flat trans-Sasakian manifold M^n , is a sub-projective manifold in the sense of Kagan.

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