Mathematical Modeling of India's Population Growth

Vikas Kumar, Shankar Lal
Department of Mathematics, H.N.B.G.U. Campus, Badshahithaul,
Tehri Garhwal-249199, Uttarakhand (India)

Abstract: The purpose of this paper is to use mathematical models to predict the population growth of India. India is the second most populous country in the world, with over 1.3 billion people, more than a sixth of the world's population. The Exponential and the Logistic growth models were applied to the model the population growth of India using data from 1981 to 2015. The data used were collected from World Development Indicator and Global Development Finance (Data-The World Bank). It was analyzed sing MATLAB software and it accurately fitted the logistic growth curve. The Exponential model predicts a growth rate of 2.3% per annum and also predicts the population to be 3535084021 in 2050. Population growth of any country depends on the vital coefficients. We determined the carrying capacity and the vital coefficients a and b are 0.03 and $5.619246017 \times 10^{-12}$ respectively. Thus the population growth of India according to the Logistic model is 3% and predicted population to be 2978550508 in 2050. We also predict that population of India will be approximately a half of its carrying capacity in the year 2042. The Mean Absolute Percentage Error (MAPE) of was computed as 6.11% for the Exponential model and 9.28% for the Logistic model.

Keywords: Exponential growth model, Logistic growth model, Population growth rate, Carrying capacity, MAPE, Vital coefficients.

1. Introduction

Population size and growth in a country directly influence the situation of economy, policy, culture, education and environment of country and determine exploring the cost of natural resources. Projection of any country's population plays a significant role in the planning as well as in the decision making for socio-economic and demographic development. Today the major issue of the world is the tremendous growth of population especially in the developing countries like India. Every government and collective sectors always require accurate idea about the future size of various entities like population, resources, demands and consumption for their planning activities. To obtain this information, the behavior of the connected variables is analyzed based on the previous data by the statisticians and mathematicians at first, and using the conclusions drawn from the analysis, they make future projection of the aimed at variable. There are enormous concern about the consequence of human population growth for social, environment and economic development. Intensifying all these problems is in the population growth. Mathematical modeling is a broad interdisciplinary science that uses mathematical and computational techniques to model and elucidate the phenomena arising in the life science. That is it is a process of mimicking reality by using the language of mathematics.

A mathematical model is a set of formulas or equation based on quantitative description or real world phenomenon and created in the hope that the behavior it predicts will resemble the real behavior on which it is based (Glenn Ledder, 2005). A population model is a type of mathematical model that is applied to the study of population dynamics. Model allow a better understanding of how complex interaction and process work. Modeling of dynamics interaction in nature can provide a manageable way of understanding how numbers change over time or in relation to each other. A model can be in many shapes, sizes and style. It is important to emphasize that a model is not real-world but merely a human construct to help us better understand real-world system. One uses model in all aspect to our life, in order to extract the important trend from complex process to permit comparison among system to facilitate analysis of cause of process action on the system and to make a prediction about the future. In this paper we model the population growth of India using Exponential and Logistic growth models.

2. Materials and Methods

A research is best understood as a process of arriving at dependent solution to the problems through the systematic collection, analysis and interpretation of data. In relation to this paper, secondary classified yearly population data of India from 1981-2015 (inclusive) were collected from world Development Indicator and Global Development Finance (Data-The World Bank). The Exponential and Logistic growth mathematical model were used to compute the projected population values and in plotting down the graph of actual and predicted population values against time in year, employing MATLAB. The Goodness of fit of the model is assessed using the Mean Absolute Percentage Error (MAPE).

3. Exponential Growth Model

In 1798 Thomas R. Malthus proposed a mathematical model of population growth. He proposed by the assumption that the population grows at a rate proportional to the size of the population. This is a reasonable assumption for a population of a bacteria or animal under ideal conditions (unlimited environment, adequate nutrition, absence of predators, and immunity from disease). Suppose we know the population N_0 at some given time $t = t_0$ and we are interested in projecting the population N at some future time $t = t_1$. In other words we want to find population function $N(t)$ for $t_0 \le t \le t_1$ satisfying $N(t_0) = N_0$.

Then considering the initial value problem

$$\frac{dN}{dt} = kN(t), t_0 \le t \le t_1;\ N(t_0) = N_0 \tag{1}$$

Integrating by variable separable in (1)

$$\int \frac{dN}{N} = k\int dt$$

$$\ln N = kt + c$$

$$N(t) = N_0 e^{kt}$$

or

$$N(t) = N_0 e^{k(t-t_0)} \tag{2}$$

Where k is a constant called Malthus factor, is multiple that determine the growth rate. Equation (1) is the Exponential growth model with (2) as its solution. It is a differential equation because it contains an unknown function N and its derivative $\frac{dN}{dt}$. Having formulated the model, we look at its consequence. If rule out a population of 0, then $N(t) > 0$ for all t. So if $k > 0$, then equation shows that $\frac{dN}{dt}$ > 0 for all t. This means that the population is always increasing. In fact, as $N(t)$ increase, equation (1) shows that $\frac{dN}{dt}$ becomes larger. In other words, the growth rate increases as the population increase. Equation (1) is appropriate for modeling population growth under ideal conditions, thus we have to recognize that a more realistic must reflect the fact a given environment has a limited resources.

4. The Logistic Growth Model

This model was proposed by the Belgian Mathematician Verhulst in the 1840s as model for world population growth. His model incorporated the idea of carrying capacity. Thus the population growth is not only on how to depend on the population size; but also on how far this size is from its upper limit i.e. its carrying capacity (maximum supportable population). He modified Malthus's model to make a population size proportional to both the previous population and a new term

$$\frac{a - bN(t)}{a} \tag{3}$$

Where a and b are vital coefficients of the population. This term depicts how far the population is from its maximum limit. Now as the population value gets closer to $\frac{a}{b}$, this new term will become very small and tends to zero, providing the right feedback to limit the population growth. Thus the second term models the competition for available resources, which tends to limit the population growth. So the modified equation using this new term is:

$$\frac{dN}{dt} = \frac{aN(t)\left(a - bN(t)\right)}{a}, \quad t_0 \le t \le t_1; \; N(t_0) = N_0 \tag{4}$$

This equation is known as the Logistic Law of population growth. Solving (4) applying the initial conditions, the (4) become

$$\frac{dN}{dt} = aN - bN^2 \tag{5}$$

Separating the variables in (5) and integrating, we obtain $\int \frac{1}{a}\left(\frac{1}{N} + \frac{b}{a - bN}\right) dN = t + c$, so that

$$\frac{1}{a}\left(\ln N - \ln\left(a - bN\right)\right) = t + c \tag{6}$$

At t = 0 and $N = N_0$ we see that $c = \frac{1}{a}\left(\ln N_0 - \ln(a - bN_0)\right)$. Equation (6) becomes

$\frac{1}{a}\left(\ln N - \ln(a-bN)\right) = t + \frac{1}{a}\left(\ln N_0 - \ln(a-bN_0)\right)$. Solving for N yields

$$N = \frac{\frac{a}{b}}{1+\left(\frac{\frac{a}{b}}{N_0}-1\right)e^{-at}} \tag{7}$$

Now taking the limit as $t \to \infty$, we get

$$N_{\max} = \lim_{t\to\infty} N = \frac{a}{b}, (a>0) \tag{8}$$

Differentiating equation (7) twice with respect to t gives

$$\frac{d^2N}{dt^2} = \frac{Ca^3e^{at}(C-e^{at})}{b(C+e^{at})^3} \tag{9}$$

Where $C = \frac{\frac{a}{b}}{N_0} - 1$

At the point of inflection this second derivative of N must be equal to zero. This will be so, when

$$C = e^{at} \tag{10}$$

Solving for t in equation (10) gives

$$t = \frac{\ln C}{a} \tag{11}$$

This is the time when the point of inflection occurs, that is, when the population is a half of the value of its carrying capacity. Let the time when the point of inflection occurs be $t = t_k$.

Suppose that at time t = 1 and t = 2, the values of N are N_1 and N_2 respectively, then from equation (7) we obtain

$$\frac{b}{a}\left(1-e^{-a}\right) = \frac{1}{N_1} - \frac{e^{-a}}{N_0} \tag{12}$$

$$\frac{b}{a}\left(1-e^{-2a}\right) = \frac{1}{N_2} - \frac{e^{-2a}}{N_0} \tag{13}$$

Dividing equation (10) by equation (9) we have

$$1+e^{-a} = \frac{\frac{1}{N_2} - \frac{e^{-2a}}{N_0}}{\frac{1}{N_1} - \frac{e^{-a}}{N_0}} \quad \text{So that} \quad e^{-a} = \frac{N_0(N_2-N_1)}{N_2(N_1-N_0)} \tag{14}$$

Putting the value of e^{-a} into the equation (9), we obtain

$$\frac{b}{a}=\frac{N_1{}^2-N_0N_2}{N_1(N_0N_1-2N_0N_2+N_1N_2)} \tag{15}$$

Therefore the limiting value of N is given by

$$N_{\max}=\lim_{t\to\infty}N=\frac{a}{b}=\frac{N_1(N_0N_1-2N_0N_2+N_1N_2)}{N_1{}^2-N_0N_2} \tag{16}$$

5. Mean Absolute Percentage Error (MAPE)

It is an evaluation statistics which is used to assess the goodness of fit of the different models in national and sub national population projections. This statistics is expressed in percentage. The concept of mean absolute percentage error (MAPE) seems to be very simple but of great importance in the selecting a parsimonious model than the other statistics. A model with smaller MAPE is preferred to the other models. The mathematical form of MAPE is given under

$$MAPE=\frac{1}{n}\sum_t^n\left|\frac{Y_t-\bar{Y}}{Y_t}\right|\times 100 \tag{17}$$

Where $Y_t, \bar{Y}$ and n are actual, fitted and number of observations of the (dependant variable) population respectively.

Lower MAPE values are better because they indicate the smaller errors are produce by the forecasting model. The following interpretation of MAPE values was suggested by Lewis (1982) as follows: Less than 10% is highly accurate forecasting, 10% to 20% is good forecasting, 21% to 50% is reasonable forecasting and 51% and above is inaccurate forecasting.

6. Results and Discussion

To estimate the future population of India, we need to determine growth rate of India's population using the Exponential Growth Model in equation (2). Using the actual population of India on table 1 below with t = 0 corresponding to the year 1981, we have $N_0=713561406$. We can solve for the growth rate k, in fact that $N_1=730303461$ when t = 1

$$730303461=713561406\times e^k$$

$$k=\ln\left(\frac{730303461}{713561406}\right)$$

$$k=0.02319165$$

Hence the general solution

$$N(t)=713561406\times e^{0.02319165t} \tag{18}$$

This suggest that the predict rate of India's population growth is 2.3% with the Exponential Growth Model. With this we projected the population of India to 2050.

Again based on table 1, Let t = 0, 1 and 2 correspond to the years 1981, 1982 and 1982 respectively. Then $N_0 = 713561406$, $N_1 = 730303461$ and $N_2 = 747374856$.

Substituting the values of N_0, N_1 and N_2 into (16) we get

$N_{\max} = \frac{a}{b} = 5338794548$. This is the predict *carrying capacity* of the population of India.

From equation (14) we obtain $e^{-a} \approx 0.97$ hence $a \approx -\ln(0.97)$

Therefore the value of a is $a \approx 0.03$. This also implies that the predicted rate of India's population is approximately 3% with the logistic Growth Model.

From $\frac{a}{b} = 5338794548$ and equation (16) we obtained $b = 5.619246017 \times 10^{-12}$.

Substituting the value of N_0, e^{-a} and $\frac{a}{b}$ in equation (7), we get

$$N(t) = \frac{5338794548}{1 + (6.48189925) \times (0.97)^t} \tag{19}$$

As the general solution and we use this to predict population of India to 2050. The predicted population of India with both models is presented on the table 1 below.

Table 1: Projection of India's population using Exponential and Logistic Growth Model

Year	Actual Population	Projected Population	
		Exponential Model	Logistic Model
1981	713561406	713561406	713561406
1982	730303461	730303460	732601968
1983	747374856	747438327	752067991
1984	764664278	764975224	771964563
1985	782085127	782923583	792296590
1986	799607235	801293059	813068783
1987	817232241	820093531	834285640
1988	834944397	839335113	855951434
1989	852736160	859028154	878070196
1990	870601776	879183245	900645698
1991	888513869	899811229	923681436
1992	906461358	920923201	947180615
1993	924475633	942530516	971146131
1994	942604211	964644796	995580556
1995	960874982	987277936	1020486119
1996	979290432	1010442110	1045864693
1997	997817250	1034149777	1071717772
1998	1016402907	1058413689	1098046460

1999	1034976626	1083246897	1124851453
2000	1053481072	1108662759	1152133021
2001	1071888190	1134674944	1179890996
2002	1090189358	1161297445	1208124754
2003	1108369577	1188544581	1236833198
2004	1126419321	1216431007	1266014749
2005	1144326293	1244971723	1295667329
2006	1162088305	1274182081	1325788348
2007	1179685631	1304077791	1356374691
2008	1197070109	1334674934	1387422709
2009	1214182182	1365989968	1418928209
2010	1230984504	1398039737	1450886439
2011	1247446011	1430841478	1483292088
2012	1263589639	1464412836	1516139271
2013	1279498874	1498771867	1549421528
2014	1295291543	1533937053	1583131817
2015	1311050527	1569927307	1617262514
2016		1606761990	1651805405
2017		1644460912	1686751694
2018		1683044351	1722091996
2019		1722533061	1757816347
2020		1762948281	1793914201
2021		1804311750	1830374445
2022		1846645717	1867185400
2023		1889972951	1904334832
2024		1934316758	1941809967
2025		1979700988	1979597498
2026		2026150054	2017683605
2027		2073688939	2056053971
2028		2122343213	2094693797
2029		2172139046	2133587823
2030		2223103221	2172720353
2031		2275263153	2212075273
2032		2328646895	2251636080
2033		2383283163	2291385903
2034		2439201343	2331307537
2035		2496431513	2371383466
2036		2555004455	2411595894
2037		2614951675	2451926778
2038		2676305416	2492357857
2039		2739098680	2532870688
2040		2803365241	2573446676

2041		2869139667	2614067110
2042		2936457337	2654713195
2043		3005354459	2695366090
2044		3075868091	2736006942
2045		3148036160	2776616919
2046		3221897486	2817177246
2047		3297491795	2857669242
2048		3374859748	2898074349
2049		3454042960	2938374173
2050		3535084021	2978550508
Mean Absolute Percentage Error		6.11%	9.28%

Source: World Development Indicator and Global Development Finance (Data-The World Bank).

Fig. 1 depicts that from 1981 the population of India has increased throughout. This may be attributed to the improvement in the education, agriculture productively, water and sanitation and health services. The exponential model predicted India's population to be 3535084021 whereas the Logistic model projected it to be 2978550508.This is presented on figure 2. From equation (17) we calculated the Mean Absolute Percentage Error (MAPE) of both models. The MAPE for Exponential and the Logistic model are 6.11% and 9.28% respectively.

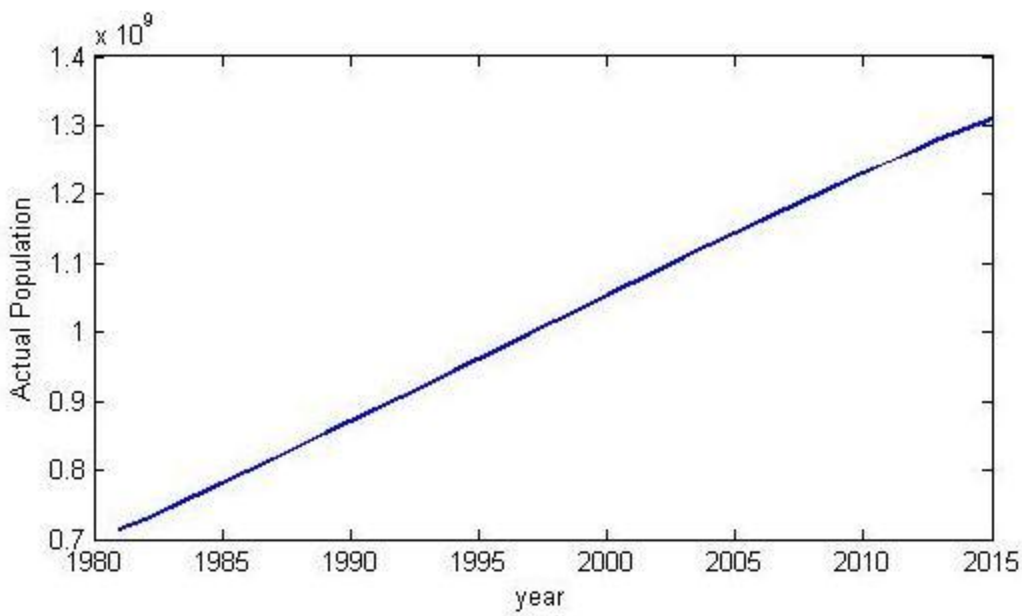

Fig. 1:Graph of actual population from 1981 to 2015

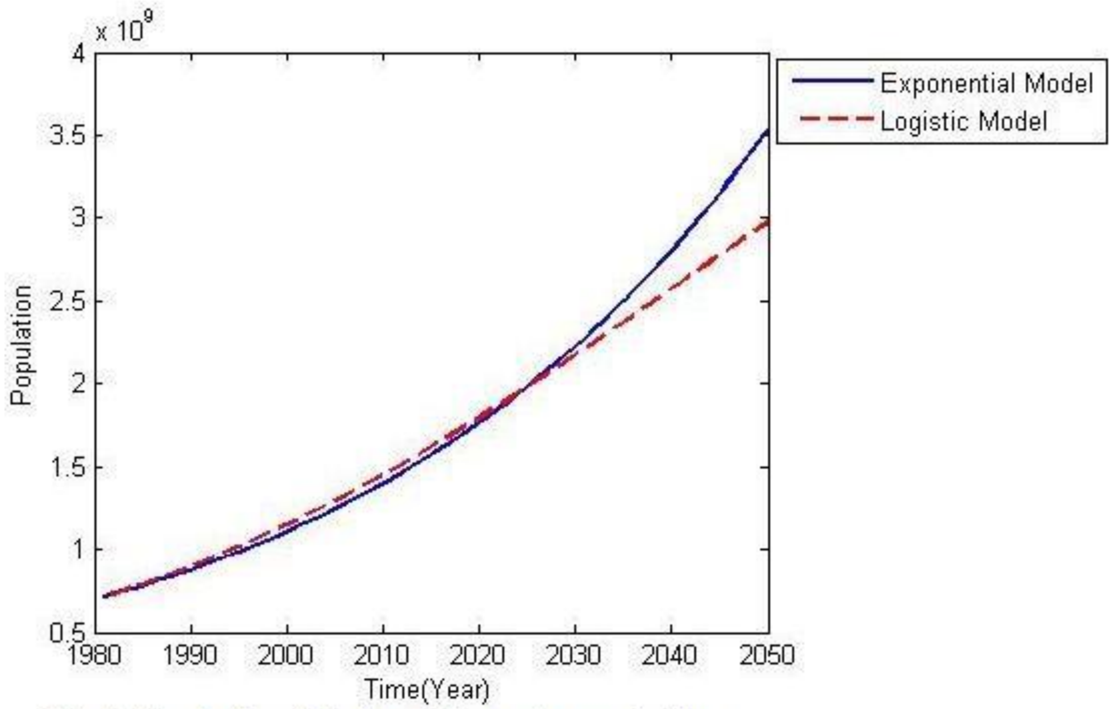

Fig. 2: Graph of predicted population values against time

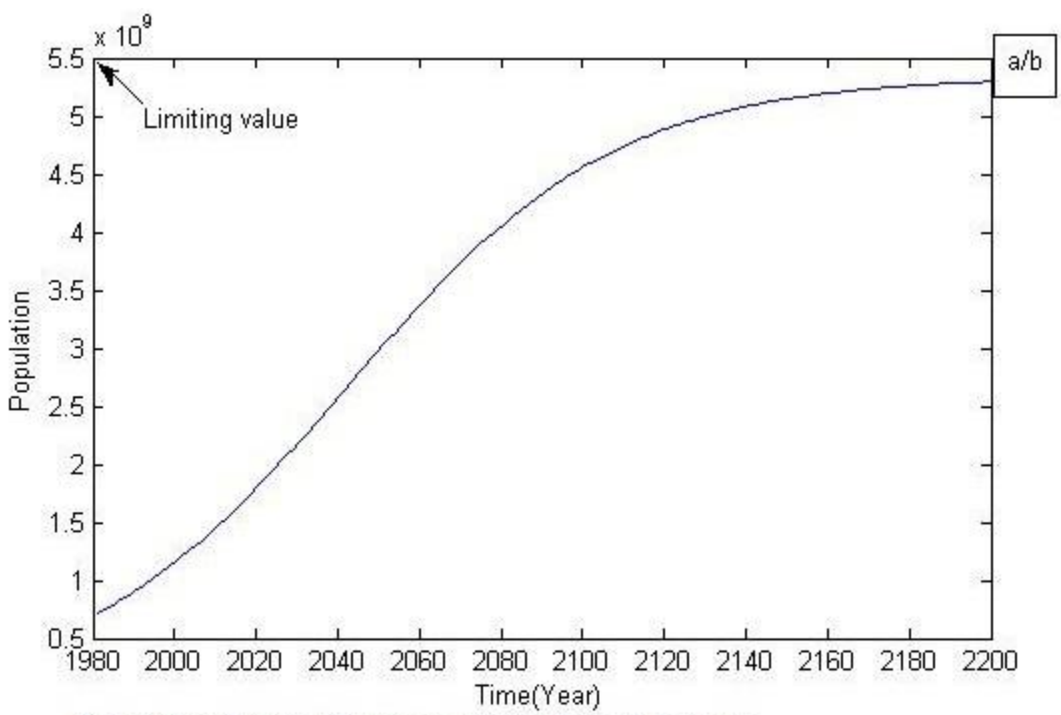

Fig. 3: Graph of predicted population values against time

Figure 2 shows the graph of the predicted population of India with both models. In figure 3, we can see that the graph of the predicted population values is an S-shaped curve. This shows that the values fitted well into the logistic curve. At first, the population starts to grow going through an exponential growth phase reaching 2654713195 (approximately a half of its carrying capacity) in the year 2042 after which the rate of growth is expected to slow down. As it gets closer to the carrying capacity, 5338794548, the growth is again expected to drastically slow down and reach a stable level.

7. Conclusion

In conclusion the Exponential Model predicted a growth rate of approximately 2.3% and predicted India's population to be 3535084021 in the year 2050 with a MAPE of 6.11%. The Logistic Model on the other hand predicted a carrying capacity for the population of India to be 5338794548. Population growth of any country depends on the vital coefficients. Here we find out that the vital coefficients a and b are 0.03 and $5.619246017 \times 10^{-12}$ respectively. Thus the population growth rate of India, according to this model, is 3% per annum. Based on this model we also find out that the population of India is expected to be 2654713195 (approximately a half of its carrying capacity) in the year 2042. It also predicted the population of India to be 2978550508 in 2050 with a MAPE of 9.28%. Based on Lewis (1982) we can conclude that the Exponential Model have a good forecasting result as compared to the Logistic Model. The following are some recommendations: Technological developments, pollution and social trends have significant influence on the vital coefficients a and b, therefore, they must be re-evaluated every few years to enhance the determination of variation in the population growth rate.

References

[1] Glen Ledder, (2005): Differential Equation: A modeling Approach. McGraw-Hill Companies Inc. USA.
[2] J.N Kapur, Q.J Khan (1979): Some Mathematical Model for Population Growth. Indian Journal for Pure Applied Math, Vol.10 (3), PP.277-286.
[3] Lewis, C.D (1982): International and business forecasting method; A practical guide to exponential smoothing and curve fitting. Butterworth Scientific, London.
[4] Population of India from World Bank webpage- http://data.worldbank.org/indicator/SP.POP.TOTL
[5] Prema Ramchandran, M. Singh, A.N. Kapoor, K.K. Lamba (Jan, 2008): "Population Growth: Trends, Projections, Challenges and Opportunities" Planning Commission, Government of India.
[6] Shilpa and Shreenidhi (2014**):** Analysis of Population Growth of India and Estimation of Future, IJIRSET, Vol. 3, Issue 9.
[7] T. R. Malthus, (1987): An Essay on The Principle Of Population (1st edition, plus excerpts 1893 2nd edition), Introduction by Philip Appleman, and assorted commentary on Malthus edited by Appleman, Norton Critical Editions, ISBN 0-393-09202-X, 1798.