# ON SOME DOUBLE INTEGRALS INVOLVING I-FUNCTION

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**ABSTRACT**

The aim of this research paper is to establish some double integrals involving I-function of two variables.

## 1. INTRODUCTION:

The I–function of two variables introduced by Sharma & Mishra [2], will be defined and represented as follows:

$$I\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] = I^{0,n:m_1,n_1:m_2,n_2}_{p_i,q_i:r:p_{i'},q_{i'}:r':p_{i''},q_{i''}:r''}\left[\begin{smallmatrix}x\\y\end{smallmatrix}\Bigg|\begin{matrix}[(a_j:\alpha_j,A_j)_{1,n}],[(a_{ji}:\alpha_{ji},A_{ji})_{n+1,p_i}]\\ [(b_{ji}:\beta_{ji},B_{ji})_{1,q_i}]\end{matrix}\right.$$
$$\left.\begin{matrix}:[(c_j;\gamma_j)_{1,n_1}],[(c_{ji'};\gamma_{ji'})_{n_1+1,p_{i'}}];[(e_j;E_j)_{1,n_2}],[(e_{ji''};E_{ji''})_{n_2+1,p_{i''}}]\\ :[(d_j;\delta_j)_{1,m_1}],[(d_{ji'};\delta\gamma_{ji'})_{m_1+1,q_{i'}}];[(f_j;F_j)_{1,m_2}],[(f_{ji''};F_{ji''})_{m_2+1,q_{i''}}]\end{matrix}\right]$$
$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)\,x^{\xi}y^{\eta}\,d\xi d\eta, \qquad (1)$$

where

$$\phi_1(\xi,\eta) = \frac{\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)}{\sum_{i=1}^{r}[\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi-A_{ji}\eta)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi+B_{ji}\eta)},$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_1}\Gamma(d_j-\delta_j\xi)\prod_{j=1}^{n_1}\Gamma(1-c_j+\gamma_j\xi)}{\sum_{i'=1}^{r'}[\prod_{j=m_1+1}^{q_{i'}}\Gamma(1-d_{ji'}+\delta_{ji'}\xi)\prod_{j=n_1+1}^{p_{i'}}\Gamma(c_{ji'}-\gamma_{ji'}\xi)},$$

$$\theta_3(\eta) = \frac{\prod_{j=1}^{m_2}\Gamma(f_j-F_j\eta)\prod_{j=1}^{n_2}\Gamma(1-e_j+E_j\eta)}{\sum_{i''=1}^{r''}[\prod_{j=m_2+1}^{q_{i''}}\Gamma(1-f_{ji''}+F_{ji''}\eta)\prod_{j=n_2+1}^{p_{i''}}\Gamma(e_{ji''}-E_{ji''}\eta)},$$

x and y are not equal to zero, and an empty product is interpreted as unity $p_i$, $p_{i'}$, $p_{i''}$, $q_i$, $q_{i'}$, $q_{i''}$, n, $n_1$, $n_2$, $n_j$ and $m_k$ are non negative integers such that $p_i \geq n \geq 0$, $p_{i'} \geq n_1 \geq 0$, $p_{i''} \geq n_2 \geq 0$, $q_i > 0$, $q_{i'} \geq 0$, $q_{i''} \geq 0$, (i = 1, …, r; i′ = 1, …, r′; i′′ = 1, …, r′′; k = 1, 2) also all the A's, $\alpha$'s, B's, $\beta$'s, $\gamma$'s, $\delta$'s, E's and F's are assumed to be positive quantities for standardization purpose; the definition of I-function of two variables given above will however, have a meaning even if some of these quantities are zero. The contour $L_1$ is in the $\xi$–plane and runs from $-\omega\infty$ to $+\omega\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(d_j-\delta_j\xi)$ (j = 1, ..........., $m_1$) lie to the right, and the poles of $\Gamma(1-c_j+\gamma_j\xi)$ (j = 1, ..., $n_1$), $\Gamma(1-a_j+\alpha_j\xi+A_j\eta)$ (j = 1, ..., n) to the left of the contour.

The contour $L_2$ is in the $\eta$–plane and runs from $-\omega\infty$ to $+\omega\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(f_j-F_j\eta)$ (j=1,....., $n_2$) lie to the right, and the poles of $\Gamma(1-e_j+E_j\eta)$ (j = 1, ..., $m_2$), $\Gamma(1-a_j+\alpha_j\xi+A_j\eta)$ (j = 1, ..., n) to the left of the contour. Also

$R' = \sum_{j=1}^{p_i} \alpha_{ji} + \sum_{j=1}^{p_i'} \gamma_{ji'} - \sum_{j=1}^{q_i} \beta_{ji} - \sum_{j=1}^{q_i'} \delta_{ji'} < 0,$

$S' = \sum_{j=1}^{p_i} A_{ji} + \sum_{j=1}^{p_i''} E_{ji''} - \sum_{j=1}^{q_i} B_{ji} - \sum_{j=1}^{q_i''} F\delta_{ji'} < 0,$

$$U' = \sum_{j=n+1}^{p_i} \alpha_{ji} - \sum_{j=1}^{q_i} \beta_{ji} + \sum_{j=1}^{m_1} \delta_j - \sum_{j=m_1+1}^{q_i'} \delta_{ji'} + \sum_{j=1}^{n_1} \gamma_j - \sum_{j=n_1+1}^{p_i'} \gamma_{ji'} > 0, \quad (2)$$

$$V' = -\sum_{j=n+1}^{p_i} A_{ji} - \sum_{j=1}^{q_i} B_{ji} - \sum_{j=1}^{m_2} F_j - \sum_{j=m_{21}+1}^{q_i''} F_{ji''} + \sum_{j=1}^{n_2} E_j - \sum_{j=n_2+1}^{p_i''} E_{ji''} > 0, \quad (3)$$

and |arg x| < ½ U'π, |arg y| < ½ V'π.

In our investigation we shall need the following results:

From Erdelyi [1, p.284, (2)]:

$$\int_{-1}^{1} (1-x)^{\rho} (1+x)^{\beta} P_n^{(\alpha,\beta)}(x)dx$$

$$= \frac{2^{\beta+\rho+1}\Gamma(\rho+1)\Gamma(\beta+n+1)\Gamma(\alpha-\rho+n)}{n!\Gamma(\alpha-\rho)\Gamma(\beta+\rho+n+2)}, \quad (4)$$

where $\operatorname{Re} \rho > -1, \operatorname{Re} \beta > -1$.

## 2. DOUBLE INTEGRALS:

In this section, we shall establish following integrals:

$$\int_{-1}^{1}\int_{-1}^{1} (1-x)^{\rho} (1+x)^{\beta} (1-y)^{\sigma} (1+y)^{k} P_m^{(\alpha,\beta)}(x) P_n^{(h,k)}(y)$$

$$.I[^{\zeta(1+x)^{\lambda}}_{\eta(1+y)^{\mu}}]dx\, dy$$

$$= \frac{2^{\beta+\rho+k+\sigma+2}\Gamma(\rho+1)\Gamma(\alpha-\rho+m)\Gamma(\sigma+1)\Gamma(h-\sigma+n)}{m!n!\Gamma(\alpha-\rho)\Gamma(h-\sigma)}$$

$$I^{0,n_1;m_2,n_2+1;m_3,n_3+1}_{p_i,q_i:r;p_i'+1,q_i'+1:r';p_i''+1,q_i''+1:r''}$$

$$[^{2^{\lambda}\zeta}_{2^{\mu}\eta}|^{......,......:(-\beta-m,\lambda),........:(-k-n,\mu),......,.......}_{......,.......:.........,(-1-\beta-\rho-m,\lambda):......,......,(-1-k-\sigma-n,\mu)}], \quad (5)$$

$\operatorname{Re} \rho + \lambda \min_{1 \le j \le m_2} [\operatorname{Re} \frac{d_j}{\delta_j}] > -1, \operatorname{Re} \beta > -1$

and $\operatorname{Re} \sigma + \mu \min_{1 \le j \le m_3} [\operatorname{Re} \frac{f_j}{F_j}] > -1, \operatorname{Re} k > -1;$

$$\int_{-1}^{1}\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\beta}(1-y)^{\sigma}(1+y)^{k}P_m^{(\alpha,\beta)}(x)P_n^{(h,k)}(y)$$

$$.I\left[{}^{\zeta(1+x)^{-\lambda}}_{\eta(1+y)^{\mu}}\right]dx\,dy$$

$$=\frac{2^{\beta+\rho+k+\sigma+2}\Gamma(\rho+1)\Gamma(\alpha-\rho+m)\Gamma(\sigma+1)\Gamma(h-\sigma+n)}{m!n!\Gamma(\alpha-\rho)\Gamma(h-\sigma)}$$

$$I^{0,n_1;m_2+1,n_2;m_3,n_3+1}_{p_i,q_i:r;p_i'+1,q_i'+1:r';p_i''+1,q_i''+1:r''}\left[{}^{2^{-\lambda}\zeta}_{2^{\mu}\eta}\Big|{}^{\ldots\ldots,\ldots\ldots:\ldots\ldots,(2+\beta+\rho+m,\lambda):(-k-n,\mu),\ldots\ldots,\ldots\ldots}_{\ldots\ldots,\ldots\ldots:(1+\beta+m,\lambda)\ldots\ldots:\ldots\ldots,\ldots\ldots,(-1-k-\sigma-n,\mu)}\right], \quad (6)$$

$$\mathrm{Re}\,\rho-\lambda\max_{1\le j\le n_2}[\mathrm{Re}\frac{c_j-1}{\gamma_j}]>-1,\mathrm{Re}\,\beta>-1$$

and $\mathrm{Re}\,\sigma+\mu\min_{1\le j\le m_3}[\mathrm{Re}\frac{f_j}{F_j}]>-1,\mathrm{Re}\,k>-1;$

**Proof of (5):**

To establish (5), expressing the I-function in the integrand as (1), changing the order of the x, y-integral and ξ, η-integral, evaluating the inner-integral with the help of (4), the value of the integral (5) is obtained. On using the same procedure as above, the integrals (6) can be established.

**PARTICULAR CASES:**

On choosing r = 1, r´ = 1 and r´´ = 1 in main integrals, we get following integrals in terms of H-function of two variables:

$$\int_{-1}^{1}\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\beta}(1-y)^{\sigma}(1+y)^{k}P_m^{(\alpha,\beta)}(x)P_n^{(h,k)}(y)$$

$$.H\left[{}^{\zeta(1+x)^{\lambda}}_{\eta(1+y)^{\mu}}\right]dx\,dy$$

$$=\frac{2^{\beta+\rho+k+\sigma+2}\Gamma(\rho+1)\Gamma(\alpha-\rho+m)\Gamma(\sigma+1)\Gamma(h-\sigma+n)}{m!n!\Gamma(\alpha-\rho)\Gamma(h-\sigma)}$$

$$H^{0,n_1;m_2,n_2+1;m_3,n_3+1}_{p_1,q_1;p_2+1,q_2+1;p_3+1,q_3+1}$$

$$\left[{}^{2^{\lambda}\zeta}_{2^{\mu}\eta}\Big|{}^{(a_j,\alpha_j;A_j)_{1,p_1}:(-\beta-m,\lambda),(c_j,\gamma_j)_{1,p_2}:\quad(-k-n,\mu),(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(-1-\beta-\rho-m,\lambda):(f_j,F_j)_{1,q_3},(-1-k-\sigma-n,\mu)}\right], \quad (7)$$

$$\mathrm{Re}\,\rho+\lambda\min_{1\le j\le m_2}[\mathrm{Re}\frac{d_j}{\delta_j}]>-1,\mathrm{Re}\,\beta>-1$$

and $\mathrm{Re}\,\sigma+\mu\min_{1\le j\le m_3}[\mathrm{Re}\frac{f_j}{F_j}]>-1,\mathrm{Re}\,k>-1;$

$$\int_{-1}^{1}\int_{-1}^{1}(1-x)^{\rho}\,(1+x)^{\beta}(1-y)^{\sigma}(1+y)^{k}P_m^{(\alpha,\beta)}(x)P_n^{(h,k)}(y)$$

$$.H\left[{}^{\zeta(1+x)^{-\lambda}}_{\eta(1+y)^{\mu}}\right]dx\,dy$$

$$=\frac{2^{\beta+\rho+k+\sigma+2}\Gamma(\rho+1)\Gamma(\alpha-\rho+m)\Gamma(\sigma+1)\Gamma(h-\sigma+n)}{m!n!\Gamma(\alpha-\rho)\Gamma(h-\sigma)}$$

$$H^{0,n_1;m_2+1,n_2;m_3,n_3+1}_{p_1,q_1;p_2+1,q_2+1;p_3+1,q_3+1}$$

$$\left[{}^{2^{-\lambda}\zeta}_{2^{\mu}\eta}\Big|{}^{(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(2+\beta+\rho+m,\lambda):\quad(-k-n,\mu),(e_j,E_j)_{1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1}:(1+\beta+m,\lambda),(d_j,\delta_j)_{1,q_2}:\quad(f_j,F_j)_{1,q_3},(-1-k-\sigma-n,\mu)}\right], \tag{8}$$

$\operatorname{Re}\rho-\lambda\max_{1\le j\le n_2}[\operatorname{Re}\frac{c_j-1}{\gamma_j}]>-1, \operatorname{Re}\beta>-1$

and $\operatorname{Re}\sigma+\mu\min_{1\le j\le m_3}[\operatorname{Re}\frac{f_j}{F_j}]>-1, \operatorname{Re}k>-1;$

and $\lambda>0, \mu>0, U>0, V>0, |\arg\zeta|<\frac{1}{2}U\pi$, where U and V are given by:

$$U=-\sum_{j=n_1+1}^{p_1}\alpha_j-\sum_{j=1}^{q_1}\beta_j+\sum_{j=1}^{m_2}\delta_j-\sum_{j=m_2+1}^{q_2}\delta_j+\sum_{j=1}^{n_2}\gamma_j-\sum_{j=n_2+1}^{p_2}\gamma_j>0, \tag{9}$$

$$V=-\sum_{j=n_1+1}^{p_1}A_j-\sum_{j=1}^{q_1}B_j+\sum_{j=1}^{m_3}F_j-\sum_{j=m_3+1}^{q_3}F_j+\sum_{j=1}^{n_3}E_j-\sum_{j=n_3+1}^{p_3}E_j>0, \tag{10}$$

## REFERENCES

1. Erdelyi, A. Tables of Integral Transform, Vol.II, McGraw-Hill, New York (1953).
2. Sharma C. K. and Mishra, P. L.: On the I-function of two variables and its certain properties, ACI, 17 (1991), 1-4.