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CYCLOTOMIC INDEX OF GRAPHS

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Abstract

Let $G = (V,E)$ be a (p,q) graph and $\varphi_p(x) = 1 + x + x^2 + ... + x^{p-1}$ be the cyclotomic polynomial associated with *G*. Then the cyclotomic index of *G* is defined as $\varphi_p(q)(\bmod p)$ and is denoted by cyl(*G*). In this paper we shall attempt to compute the cyclotomic index of some special type of graphs. We also establish some fundamental properties of cyclotomic index of certain graphs.

Keywords

Cyclotomic polynomial of graphs, cyclotomic index of graphs .

Introduction

By a graph $G = (V,E)$ we mean a finite undirected graph with neither loops nor multiple edges. The order and size of the graph are denoted by *p* and *q* respectively. For various graph theoretic notations and terminology we follow F. Harrary [1] and for basic number theoretic results we refer [4].

Definition 1.1 Let *G* be a (p,q) graph. Consider the p^{th} cyclotomic polynomial associated with *G* given by $\varphi_p(x) = 1 + x + x^2 + ... + x^{p-1}$. Then the cyclotomic index of *G* defined as $\varphi_p(q)(\bmod p)$ and is denoted by $cyl(G)$.

Every graph G has a cyclotomic polynomial and every cyclotomic polynomial is of some simple graph G. But There is infinite number of graph for a given cyclotomic polynomial.

The following are some simple observations which follow immediately from the definition of a cyclotomic polynomial.

Observation 1.2 G be a graph of order n then $cyl(G) < n$.

Observation 1.3 Let G_1, G_2 be two isomorphic graph then $cyl(G_1) = cyl(G_2)$.

Observation 1.4 The cyclotomic polynomial of trivial graphs with n vertices is $\varphi_n(x) = 1 + x + x^2 + ... + x^{n-1}$ and $cyl(G) = \varphi_n(0)(\bmod n) = 1$.

Observation 1.5 The cyclotomic polynomial of a (p,q) graphs G with $p = n, q = n+1$ is $\varphi_n(x) = 1 + x + x^2 + ... + x^{n-1}$ and $cyl(G) = 0$.

Observation 1.6 The cyclotomic polynomial of a (p,q) graphs G with $p = n, q = 1$ is $\varphi_n(x) = 1 + x + x^2 + ... + x^{n-1}$ and $cyl(G) = 0$.

Observation 1.7 The cyclotomic polynomial of a (p,q) graphs G with $p = n, q = n-1$ is $\varphi_n(x) = 1 + x + x^2 + ... + x^{n-1}$ and $cyl(G) = \left\{\begin{array}{ll} 0 & \text{if n is even} \\ 1 & \text{if n is odd} \end{array}\right\}$.

Observation 1.8 The star graph K_{1n}, $cyl(K_{1n}) = \left\{\begin{array}{ll} 0 & \text{if n is odd} \\ 1 & \text{if n is even} \end{array}\right\}$.

1. Main Results

Theorem 2.1 : For any tree T with order n, $cyl(T) = \left\{\begin{array}{ll} 0 & \text{if n is even} \\ 1 & \text{if n is odd} \end{array}\right\}$

Proof : If T is a tree with n vertices then it has precisely $n-1$ edges. Then the cyclotomic polynomial $\varphi_n(x)=1+x+x^2+...+x^{n-1}$ and

$$\begin{aligned} cyl(T) &= 1+(n-1)+(n-1)^2+...+(n-1)^{n-1}(\text{mod } n) \\ &= 1+(-1)+(-1)^2+...+(-1)^{n-1}(\text{mod } n) \\ &= \begin{Bmatrix} 0 & \text{if n is even} \\ 1 & \text{if n is odd} \end{Bmatrix} \end{aligned}$$

Hence the proof.

Theorem 2.2 : Let *G* be connected (p,q) graph which is unicyclic then , $cyl(G)=1$.

Proof : *G* be connected graph with p vertices and it is unicyclic then it has precisely p edges. Then the cyclotomic polynomial $\varphi_p(x)=1+x+x^2+...+x^{p-1}$ and

$$\begin{aligned} cyl(G) &= 1+p+p^2+...+p^{p-1}(\text{mod } p) \\ &= 1(\text{mod } p) \\ &= 1 \end{aligned}$$

Theorem 2.3 : There is no graph *G* of 3 vertices with $cyl(G)=2$.

Proof : **:** Let *G* be a $(3,q)$ graph. Then the cyclotomic polynomial $\varphi_3(x)=1+x+x^2$ and

$cyl(G)=1+q+q^2(\text{mod } 3)$.

Now consider q is of the form $3n+r$,where $0\le r\le 2$.

If $q=3n$

$$\begin{aligned} cyl(G) &= 1+q+q^2(\text{mod } 3) \\ &= 1+3n+(3n)^2(\text{mod } 3) \\ &= 1 \end{aligned}$$

If $q=3n+1$

$$\begin{aligned} cyl(G) &= 1+q+q^2 \pmod 3 \\ &= 1+3n+1+(3n+1)^2 \pmod 3 \\ &= 1+1+1 \pmod 3 \\ &= 0 \end{aligned}$$

If $q = 3n+2$

$$\begin{aligned} cyl(G) &= 1+q+q^2 \pmod 3 \\ &= 1+3n+2+(3n+2)^2 \pmod 3 \\ &= 1+2+4 \pmod 3 \\ &= 1 \end{aligned}$$

So the possible $cyl(G)$ are o and 1. For any graph of 3 vertices $cyl(G) = 0$ or 1.

Theorem 2.3 : There is no graph *G* of 4 vertices with $cyl(G) = 2$.

Proof : : Let *G* be a $(4,q)$ graph. Then the cyclotomic polynomial $\varphi_4(x) = 1+x+x^2+x^3$ and

$cyl(G) = 1+q+q^2+q^3 \pmod 4$.

Now consider q is of the form $4n+r$,where $0 \le r \le 3$.

If $q = 4n$

$$\begin{aligned} cyl(G) &= 1+q+q^2+q^3 \pmod 4 \\ &= 1+4n+(4n)^2+(4n)^3 \pmod 4 \\ &= 1 \end{aligned}$$

If $q = 4n+1$

$$\begin{aligned} cyl(G) &= 1+q+q^2+q^3 \pmod 4 \\ &= 1+4n+1+(4n+1)^2+(4n+1)^3 \pmod 4 \\ &= 1+1+1+1 \pmod 4 \\ &= 0 \end{aligned}$$

If $q = 4n+2$

$$\begin{aligned} cyl(G) &= 1+q+q^2+q^3 (\text{mod } 4) \\ &= 1+4n+2+(4n+2)^2+(4n+2)^3 (\text{mod } 4) \\ &= 1+2+4+8 (\text{mod } 4) \\ &= 3 \end{aligned}$$

If $q = 4n+3$

$$\begin{aligned} cyl(G) &= 1+q+q^2+q^3 (\text{mod } 4) \\ &= 1+4n+3+(4n+3)^2+(4n+3)^3 (\text{mod } 4) \\ &= 1+3+9+27 (\text{mod } 4) \\ &= 0 \end{aligned}$$

So the possible $cyl(G)$ are $0,\ 1$ and 3. For any graph of 4 vertices $cyl(G) = 0,\ 1$ or 3.

Theorem 2.4 : There is no graph *G* of 5 vertices with $cyl(G) = 2, 3$ and 4.

Proof : **:** Let *G* be a $(5, q)$ graph. Then the cyclotomic polynomial $\varphi_5(x) = 1+x+x^2+x^3+x^4$ and

$cyl(G) = 1+q+q^2+q^3+q^4 (\text{mod } 5)$.

Now consider q is of the form $5n+r$,where $0 \le r \le 4$.

If $q = 5n$

$$\begin{aligned} cyl(G) &= 1+q+q^2+q^3+q^4 (\text{mod } 5) \\ &= 1+5n+(5n)^2+(5n)^3+(5n)^4 (\text{mod } 5) \\ &= 1 \end{aligned}$$

If $q = 5n+1$

$$cyl(G) = 1+q+q^2+q^3+q^4 \pmod{5}$$
$$= 1+5n+1+(5n+1)^2+(5n+1)^3+(5n+1)^3 \pmod{5}$$
$$= 1+1+1+1+1 \pmod{5}$$
$$= 0$$

If $q = 5n+2$

$$cyl(G) = 1+q+q^2+q^3+q^4 \pmod{5}$$
$$= 1+5n+2+(5n+2)^2+(5n+2)^3+(5n+2)^4 \pmod{5}$$
$$= 1+2+4+8+2^4 \pmod{5}$$
$$= 1$$

If $q = 5n+3$

$$cyl(G) = 1+q+q^2+q^3+q^4 \pmod{5}$$
$$= 1+5n+3+(5n+3)^2+(5n+3)^3+(5n+3)^4 \pmod{5}$$
$$= 1+3+9+27+3^4 \pmod{5}$$
$$= 1$$

If $q = 5n+4$

$$cyl(G) = 1+q+q^2+q^3+q^4 \pmod{5}$$
$$= 1+5n+4+(5n+4)^2+(5n+4)^3+(5n+4)^4 \pmod{5}$$
$$= 1+4+4^2+4^3+4^4 \pmod{5}$$
$$= 1$$

So the possible $cyl(G)$ are 0 and 1. For any graph of 5 vertices $cyl(G) = 0$ or 1.

Theorem 2.5 : There is no graph *G* of $2p$ vertices with $cyl(G) = 2$, where p is a prime number .

Proof : **:** Let *G* be a (n,q) graph. $n = 2p$. Then the cyclotomic polynomial

$\varphi_n(x) = 1+x+x^2+...+x^{n-1}$ and

$cyl(G) = 1+q+q^2+...+q^{n-1} \pmod{n}$.

If $q \equiv 1(\bmod n)$ then $cyl(G) = 0$

If $q \neq 1(\bmod n)$ then

$$cyl(G) = 1 + q + q^2 + \ldots + q^{n-1} (\text{mod n})$$
$$= \frac{q^n - 1}{q - 1} (\text{mod n})$$

By Euler theorem $q^{\phi(n)} \equiv 1(\bmod n)$ since $\gcd(q, n) = 1$.

$\phi(n) = \phi(2p) = p - 1$

So $q^{p-1} \equiv 1(\bmod n)$ that is $q^p \equiv q(\bmod n)$ then

$q^n = q^{2p} \equiv q^2 (\bmod n)$

$$cyl(G) = \frac{q^n - 1}{q - 1} (\bmod n) \equiv \frac{q^2 - 1}{q - 1} (\bmod n)$$
$$\equiv q + 1(\bmod n)$$

Since $q \neq 1(\bmod n)$, $cyl(G) = q + 1$ cannot be 2.

Theorem 2.6 : Let *G is* (p, q) graph. If p is a prime number then $cyl(G) = 1$, provided $q \neq 1(\bmod p)$.

Proof : : Let *G* be a (p, q) .Then the cyclotomic polynomial

$\varphi_p(x) = 1 + x + x^2 + \ldots + x^{p-1}$ and

$$cyl(G) = 1 + q + q^2 + \ldots + q^{p-1} (\bmod p)$$
$$= \frac{q^p - 1}{q - 1} (\bmod p)$$

Now consider q is of the form $np + r$,where $0 \le r < p$.

$$q^p = (np + r)^p \equiv r^p \bmod p$$

Since $0 \le r < p$ and p is prime , $\gcd(r, p) = 1$

$\therefore \quad r^p \equiv r (\mathrm{mod}\ p)$ so

$$q^p \equiv r (\mathrm{mod}\ p)$$

$$\begin{aligned} cyl(G) &= \frac{q^p - 1}{q-1} (\mathrm{mod}\ p) \\ &= \frac{r-1}{q-1} (\mathrm{mod}\ p) \qquad [\because \mathrm{q} \equiv \mathrm{r} \bmod p] \\ &= \frac{r-1}{r-1} = 1 (\mathrm{mod}\ p) \end{aligned}$$

Corollary 2.5 : There is no graph *G* of p vertices with $cyl(G) = 2, 3, \ldots p-1$. where p is a prime number .

Problem 2.6 : The Cycle C_n, $cyl(C_n) = 1$.

Problem 2.7 : The Path P_n, $cyl(P_n) = \begin{Bmatrix} 0 & \text{if n is even} \\ 1 & \text{if n is odd} \end{Bmatrix}$.

Problem 2.8 : The Complete graph K_n, $cyl(K_n) = \begin{Bmatrix} \frac{n}{2}+1 & \text{if n is even} \\ 1 & \text{if n is odd} \end{Bmatrix}$.

Problem 2.9: The regular graph G of order n, $cyl(G) = \begin{Bmatrix} \frac{n}{2}+1 & \text{if k is odd} \\ 1 & \text{if k is even} \end{Bmatrix}$.

Problem 2.10 : The Wheel W_{n+1}, $cyl(W_{n+1}) = \begin{Bmatrix} 1 & \text{if n is even} \\ 3 & \text{if n is odd} \end{Bmatrix}$.

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