MAXIMUM MODULUS OF POLYNOMIALS HAVING SOME ZEROS AT ORIGIN

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Abstract: Let $p(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n. Concerning the estimate for the maximum modulus of a polynomial on the circle $|z|=R$, $R>0$, in terms of its degree and the maximum modulus on the unit circle, we have several well known results for the case $R\geq 1$ and $r\leq 1$ respectively. In this paper, we have obtained bounds for the maximum modulus of polynomials having some zeros in the interior of a circle of radius $R\geq 1$. Our result improves as well as generalizes the bounds obtained by other authors for the same class of polynomials.

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let $p(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n. Concerning the estimate for the maximum modulus of a polynomial on the circle $|z|=R$, $R>0$, in terms of its degree and the maximum modulus on the unit circle, we have the following well known results.

Theorem A *If* $p(z)$ *is a polynomial of degree n, then for every* $R\geq 1$,

$$\max_{|z|=R}|p(z)|\leq R^n\max_{|z|=1}|p(z)| . \tag{1.1.}$$

The result is best possible and extremal polynomial is $p(z)=\lambda z^n$, $\lambda(\neq 0)$ *being a complex number.*

Inequality (1.1.) is a simple deduction from the maximum modulus principle (for reference see [7] or [6]).

For the case $r\leq 1$ we have the following result.

Theorem B. *If $p(z)$ is a polynomial of degree n, then for $r \le 1$,*

$$\max_{|z|=r}|p(z)| \ge r^n \max_{|z|=1}|p(z)|. \qquad (1.2.)$$

The result is best possible and extremal polynomial is $p(z)=\lambda z^n$, $\lambda(\ne 0)$ being a complex number.

Inequality (1.2.) is due to Zarantonello and Varga [9].

Theorem C. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z|<1$, then for $r \le 1$,*

$$\max_{|z|=r}|p(z)| \ge \left(\frac{1+r}{2}\right)^n \max_{|z|=1}|p(z)|. \qquad (1.3.)$$

The result is best possible and equality in inequality (1.3) *holds for* $p(z)=\left(\frac{1+z}{2}\right)^n$.

The inequality (1.1) is due to Ankeny and Rivlin [1] and inequality (1.3) is due to Rivlin [8].

For the case $0<\rho\le 1$, we have the following result due to Aziz [2].

Theorem D. *Let $p(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n, which does not vanish in $|z|<k$, $k \ge 1$. Then for $0<\rho\le 1$*

$$\max_{|z|=\rho}|p(z)| \ge \left(\frac{\rho+k}{1+k}\right)^n \max_{|z|=1}|p(z)|. \qquad (1.4)$$

The result is sharp and equality in (1.4) *is attained for* $p(z)=c(ze^{i\beta}+k)^n$, $c(\ne 0)\in C$ *and* $\beta \in R$.

The following result is due to Jain [5].

Theorem E. *If $p(z)$ be a polynomial of degree n, having all its zeros in $|z|\le k$, $k>1$, then for $k<R<k^2$,*

$$\max_{|z|=R}|p(z)| \geq R^s\left(\frac{R+k}{1+k}\right)\max_{|z|=1}|p(z)|. \tag{1.5}$$

where $s(<n)$ *is the order of a possible zero of* $p(z)$ *at origin.*

In this paper, we prove the following generalization of Theorem E by involving the coefficients of the polynomial $p(z)=\sum_{j=0}^{n}a_j z^j$. In fact we prove the following

Theorem 1. *Let* $p(z)=\sum_{j=0}^{n}a_j z^j$ *be a polynomial of degree n, having all its zeros in* $|z|\leq k$, $k>1$, *then for* $k<R<k^2$,

$$\max_{|z|=R}|p(z)| \geq R^s\left\{\frac{(R^{n-s-1}k^2+R^{n-s+1})(n-s)|a_n|+2R^{n-s}|a_{n-1}|}{(R^{n-s-1}k^2+R)(n-s)|a_n|+(R^{n-s}+1)|a_{n-1}|}\right\}\max_{|z|=1}|p(z)|$$
$$+\frac{R^s}{k^s}\left\{\frac{(R^{n-s}-1)(|a_{n-1}|+|a_n|(n-s)R)}{(n-s)|a_n|(R^{n-s-1}k^2+R)+(R^{n-s}+1)|a_{n-1}|}\right\}\min_{|z|=k}|p(z)|, \tag{1.6}$$

where s is the order of a possible zero of $p(z)$ *at the origin .*

2. LEMMAS.

For the proof of the above theorems, we need the following lemmas.

Lemma 2.1. *If* $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ *is a polynomial of degree n, having no zeros in* $|z|<k$, $k\geq 1$,

then $$\max_{|z|=1}|p'(z)| \leq n\frac{n|a_0|+k^2|a_1|}{(1+k^2)n|a_0|+2k^2|a_1|}\max_{|z|=1}|p(z)|. \tag{2.1}$$

The above lemma is due to Govil, Rahman and Schmeisser [4].

The above lemma is due to Dewan, Singh and Yadav [3].

Lemma 2.2. *If* $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ *has no zeros in* $|z|<k$, $k\geq 1$, *then*

$$\max_{|z|=1}|p'(z)| \le n\left(\frac{n|a_0|+k^2|a_1|}{(1+k^2)n|a_0|+2k^2|a_1|}\right)\max_{|z|=1}|p(z)|$$

$$-\frac{n}{k^{n-2}}\left(\frac{n|a_0|+|a_1|)}{(1+k^2)n|a_0|+2k^2|a_1|}\right)\min_{|z|=k}|p(z)|. \qquad (2.2)$$

Lemma 2.3. *If* $p(z)=\sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n having all its zeros in* $|z|\ge k$, $k>0$, *then for* $r \le k \le R$, *we have*

$$\max_{|z|=r}|p(z)| \ge \frac{r^{n-1}(r^2+k^2)n|a_0|+2k^2|a_1|}{n|a_0|r(k^2r^{n-2}+R^n)+k^2|a_1|(r^n+R^n)}\max_{|z|=R}$$

$$+\left\{\frac{r^{n-1}(R^n-r^n)(n|a_0+r|a_1||)}{k^{n-2}\left[n|a_0|r(k^2r^{n-2}+R^n)+k^2|a_1|(r^n+R^n)\right]}\right\}\min_{|z|=k}|p(z)|. \qquad (2.3)$$

Proof of Lemma 2.3. Since $p(z)$ does not vanish in $|z|<k$, $k\ge 1$, the polynomial $T(z)=p(rz)$ does not vanish in $|z|<\frac{k}{r}, \frac{k}{r}\ge 1$, therefore applying Lemma 2.2 to $T(z)$, we get

$$\max_{|z|=1}|T'(z)| \le n\left\{\frac{n|a_0|+\frac{k^2}{r^2}r|a_1|)}{(1+\frac{k^2}{r^2})n|a_0|+2\frac{k^2}{r^2}r|a_1|}\right\}\max_{|z|=1}|T(z)|$$

$$-\frac{n}{\left(\frac{k}{r}\right)^{n-2}}\left\{\frac{n|a_0|+r|a_1|)}{(1+\frac{k^2}{r^2})n|a_0|+2\frac{k^2}{r^2}r|a_1|}\right\}\min_{|z|=\frac{k}{r}}|T(z)|$$

or

$$\max_{|z|=r} r\left|p'(rz)\right| \le n r\left\{\frac{n|a_0|r + k^2|a_1|)}{(r^2+k^2)n|a_0| + 2k^2 r|a_1|}\right\}\max_{|z|=1}|p(z)|$$

$$-\frac{n r^n}{k^{n-2}}\left\{\frac{n|a_0| + r|a_1|)}{(r^2+k^2)n|a_0| + 2k^2 r|a_1|}\right\}\min_{|z|=\frac{k}{r}}|p(rz)|$$

which is equivalent to

$$\max_{|z|=r}|p'(z)| \le n\left\{\frac{n|a_0|r + k^2|a_1|)}{(r^2+k^2)n|a_0| + 2k^2 r|a_1|}\right\}M(p,r)$$

$$-\frac{n r^{n-1}}{k^{n-2}}\left\{\frac{n|a_0| + r|a_1|)}{(r^2+k^2)n|a_0| + 2k^2 r|a_1|}\right\}m(p,k). \qquad (2.4)$$

Again as $p'(z)$ is a polynomial of degree $n-1$, by maximum modulus principle [6, p. 158, problem III 269], we have

$$\frac{M(p',t)}{t^{n-1}} \le \frac{M(p',r)}{r^{n-1}}, \qquad \text{for } t \ge r \qquad (2.5)$$

Combining inequalities (2.4) and (2.5), we have

$$\max_{|z|=t}|p'(z)| \le \frac{n t^{n-1}}{r^{n-1}}\left[\begin{array}{l}\left\{\frac{n|a_0|r + k^2|a_1|)}{(r^2+k^2)n|a_0| + 2k^2 r|a_1|}\right\}M(p,r) \\ -\frac{r^{n-1}}{k^{n-2}}\left\{\frac{n|a_0| + r|a_1|)}{(r^2+k^2)n|a_0| + 2k^2 r|a_1|}\right\}m(p,k)\end{array}\right].$$

Now, for $0 \le \theta < 2\pi$, we have

$$\left|p(\mathrm{Re}^{i\theta}) - p(re^{i\theta})\right| \leq \int_r^R \left|p'(te^{i\theta})\right| dt$$

$$\leq \int_r^R \frac{nt^{n-1}}{r^{n-1}} \left[\left\{ \frac{n|a_0|r + k^2|a_1|)}{(r^2+k^2)n|a_0| + 2k^2 r|a_1|} \right\} M(p,r) - \frac{r^{n-1}}{k^{n-2}} \left\{ \frac{n|a_0| + r|a_1|)}{(r^2+k^2)n|a_0| + 2k^2 r|a_1|} \right\} m(p,k) \right] dt$$

$$= \frac{R^n - r^n}{r^{n-1}} \left[\left\{ \frac{n|a_0|r + k^2|a_1|)}{(r^2+k^2)n|a_0| + 2k^2 r|a_1|} \right\} M(p,r) - \frac{r^{n-1}}{k^{n-2}} \left\{ \frac{n|a_0| + r|a_1|)}{(r^2+k^2)n|a_0| + 2k^2 r|a_1|} \right\} m(p,k) \right].$$

This is equivalent to

$$M(p,R) \leq \frac{r^{n-1}[(r^2+k^2)n|a_0| + 2k^2 r|a_1|] + (R^n - r^n)[n|a_0|r + k^2|a_1|]}{r^{n-1}\{(r^2+k^2)n|a_0| + 2k^2 r|a_1|\}} M(p,r) - \frac{R^n - r^n}{k^{n-2}} \left\{ \frac{n|a_0| + r|a_1|)}{(r^2+k^2)n|a_0| + 2k^2 r|a_1|} \right\} m(p,k).$$

From which the proof of Lemma 2.3 follows.

3. PROOF OF THE MAIN THEOREM

Proof of the Theorem 1. The polynomial $p(z)$ of degree n has all its zeros in $|z| \leq k$, $k > 1$, with s-fold zeros at the origin, implies that the polynomial $q(z) = z^n \overline{p(1/\bar{z})}$ is of degree (n-s) and has all its zeros in $|z| \geq \frac{1}{k}$, $\frac{1}{k} < 1$.

On applying Lemma 2.3 to the polynomial $q(z)$ with $R = 1$, we obtain for $\frac{1}{k^2} < r < \frac{1}{k}$,

$$\max_{|z|=r}|q(z)| \geq \frac{r^{n-s-1}\left(r^2+\frac{1}{k^2}\right)(n-s)|\overline{a_n}|+2\frac{1}{k^2}r^{n-s}|\overline{a_{n-1}}|}{(n-s)|\overline{a_n}|r\left(\frac{1}{k^2}r^{n-s-2}+1\right)+\frac{1}{k^2}|\overline{a_{n-1}}|\left(r^{n-s}+1\right)}\max_{|z|=1}|q(z)|$$

$$+\frac{r^{n-s-1}(1-r^{n-s})((n-s)|\overline{a_n}|+r|\overline{a_{n-1}}|)}{\frac{1}{k^{n-s-2}}\left\{(n-s)|\overline{a_n}|r\left(\frac{1}{k^2}r^{n-s-2}+1\right)+\frac{1}{k^2}|\overline{a_{n-1}}|\left(1+r^{n-s}\right)\right\}}\min_{|z|=\frac{1}{k}}|p(z)|$$

or equivalently

$$\max_{|z|=r}\left|z^n\overline{p\left(\frac{1}{\bar{z}}\right)}\right| \geq \frac{r^{n-s-1}\left\{\left(r^2+\frac{1}{k^2}\right)(n-s)|a_n|+\frac{2r}{k^2}|a_{n-1}|\right\}}{(n-s)|a_n|r\left(\frac{r^{n-s-2}}{k^2}+1\right)+\frac{|a_{n-1}|}{k^2}\left(r^{n-s}+1\right)}\max_{|z|=1}|p(z)|$$

$$+\frac{r^{n-s-1}k^{n-s-2}(1-r^{n-s})((n-s)|a_n|+r|a_{n-1}|)}{\left\{(n-s)|a_n|r\left(\frac{r^{n-s-2}}{k^2}+1\right)+\frac{|a_{n-1}|\left(1+r^{n-s}\right)}{k^2}\right\}}\min_{|z|=\frac{1}{k}}\left|z^n\overline{p\left(\frac{1}{\bar{z}}\right)}\right|$$

$$\text{for } \frac{1}{k^2}<r<\frac{1}{k}.$$

The above inequality is equivalent to

$$\max_{|z|=\frac{1}{r}}|p(z)| \geq \frac{r^{-s-1}\left\{\left(r^2+\frac{1}{k^2}\right)(n-s)|a_n|+\frac{2r}{k^2}|a_{n-1}|\right\}}{(n-s)|a_n|r\left(\frac{r^{n-s-2}}{k^2}+1\right)+\frac{|a_{n-1}|}{k^2}\left(r^{n-s}+1\right)}\max_{|z|=1}|p(z)|$$

$$+\frac{r^{-s-1}k^{n-s-2}(1-r^{n-s})((n-s)|a_n|+r|a_{n-1}|)}{\left\{(n-s)|a_n|r\left(\frac{r^{n-s-2}}{k^2}+1\right)+\frac{|a_{n-1}|\left(1+r^{n-s}\right)}{k^2}\right\}}\frac{1}{k^n}\min_{|z|=k}|p(z)| \tag{3.1}$$

for $\frac{1}{k^2} < r < \frac{1}{k}$.

Now replacing r by $\frac{1}{R}$ we get from inequality (3.1)

$$\max_{|z|=R}|p(z)| \geq \frac{R^{s+1}\left\{\left(\frac{1}{R^2}+\frac{1}{k^2}\right)(n-s)|a_n| + \frac{2}{k^2R}|a_{n-1}|\right\}}{(n-s)|a_n|\frac{1}{R}\left(\frac{1}{k^2R^{n-s-2}}+1\right)+\frac{|a_{n-1}|}{k^2}\left(\frac{1}{R^{n-s}}+1\right)}\max_{|z|=1}|p(z)|$$
$$+\frac{R^{s+1}k^{n-s-2}\left(1-\frac{1}{R^{n-s}}\right)\left((n-s)|a_n|+\frac{|a_{n-1}|}{R}\right)}{(n-s)|a_n|\frac{1}{R}\left(\frac{1}{k^2R^{n-s-2}}+1\right)+\frac{|a_{n-1}|}{k^2}\left(1+\frac{1}{R^{n-s}}\right)}\frac{1}{k^n}\min_{|z|=k}|p(z)|$$

for $k < R < k^2$.

The above inequality on simplification reduces to

$$\max_{|z|=R}|p(z)| \geq R^s\left\{\frac{(R^{n-s-1}k^2+R^{n-s+1})(n-s)|a_n|+2R^{n-s}|a_{n-1}|}{(R^{n-s-1}k^2+R)(n-s)|a_n|+(R^{n-s}+1)|a_{n-1}|}\right\}\max_{|z|=1}|p(z)|$$
$$+\frac{R^s}{k^s}\left\{\frac{(R^{n-s}-1)(|a_{n-1}|+|a_n|(n-s)R)}{(n-s)|a_n|(R^{n-s-1}k^2+R)+(R^{n-s}+1)|a_{n-1}|}\right\}\min_{|z|=k}|p(z)|,$$

for $k < R < k^2$.

This completes the proof of Theorem 1.

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