Best Approximation Results using Property (E.A.) and (CLR) in Complex Valued Metric Spaces

Savita Rathee [a], Reetu [b]
[a] Department of Mathematics, M.D. University, Rohtak
[b]Vaish College, Rohtak (Haryana)-124001, India

Abstract:

In this paper we establish some best approximation results with the help of existing common fixed point theorems using the property (E.A.) and (CLR) in the setting of complex valued metric spaces. The results proved in this paper extend and generalize various known results of complex valued metric spaces.

Keywords: Complex valued metric space; Common fixed point; Property (E.A); Property (CLR); Weakly Compatible maps; Best approximation.

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1. Introduction:

Banach fixed point theorem is one of the pivotal result in nonlinear analysis known as Banach's Contraction Principle. This principal is constructive in nature which explains the existence and uniqueness of fixed points of operators or mappings. This Principle has been obtained in several directions like 2-metric spaces, D-metric spaces, G-metric spaces etc. (see [2,3,4,6,7,8]). These generalizations were made either by weakening the contractive condition or by imposing some additional conditions on ambient space.

Azam et al. [1] introduced the concept of complex-valued metric space space which is more general than well-known metric spaces and obtained fixed point theorems of contractive type mappings using the rational inequality in a complex-valued metric space. In 2013, Verma and Pathak [9] defined the concept of property (E.A) in a complex-valued metric space and gave following common fixed point result for two pairs of weakly compatible mappings:

Theorem 1.1:. Let (X, d) be a complex valued metric space and $A, B, S, T: X \to X$ be four self-maps satisfying

(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$,

(ii) $d(Ax, By) \precsim k \max\{d(Sx, Ty), d(By, Sx), d(By, Ty)\}$ for all $x, y \in X$ and $0 < k < 1$, (1.1)

(iii) the pairs (A, S) and (B, T) are weakly compatible,

(iv) one of the pairs (A, S) or (B, T) satisfies the property (E.A).

If the range of one of the mappings S(X) or T(X) is a complete subspace of X, then the mappings A, B, S and T have a unique common fixed point in X.

Further, they proved a common fixed point theorem for two pairs of self mappings satisfying the common limit property in the range of a mapping called (CLR)-property by Sintunavarat and Kumam in the following sense:

Theorem 1.2: Let (X, d) be a complex valued metric space and A, B, S, T : X → X be four self mappings satisfying

(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$,

(ii) $d(Ax, By) \preccurlyeq k \max\{d(Sx, Ty), d(By, Sx), d(By, Ty)\}$ for all x, y ∈ X and 0 < k < 1, (1.2)

(iii) the pairs (A, S) and (B, T) are weakly compatible.

If the pair (A, S) satisfies (CLR_A) property or the pair (B, T) satisfies (CLR_B) property, then the mappings A, B, S and T have a unique common fixed point in X.

Afterwards, Kumar et. al.[5] proved the following common fixed point theorem without considering the completeness of space, continuity of maps and using the properties (E.A.) and (CLR):

Theorem 1.3: Let (X, d) be a complex valued metric space. Suppose that f, g, h and k are four self-maps of X satisfying the following conditions:

(i) $f(X) \subseteq h(X)$, $g(X) \subseteq k(X)$,

(ii) $d(fx, gy) \preccurlyeq \lambda \max\{d(kx, hy), d(kx, fx), d(hy, gy), \frac{d(kx, gy)+d(hy, fx)}{2}\}$ (1.3)

(iii) one of the pairs (f, k) or (g, h) satisfies the property (E.A),

(iv) the pairs (f, k) and (g, h) are weakly compatible.

If k(X) or h(X) is a closed subset of X, then f, g, h and k have a unique common fixed point.

Theorem 1.4: Let (X, d) be a complex valued metric space. Suppose that f, g, h and k are four self-maps of X satisfying the following conditions:

(i) $f(X) \subseteq h(X)$, $g(X) \subseteq k(X)$,

(ii) $d(fx, gy) \preccurlyeq \lambda \max\{d(kx, hy), d(kx, fx), d(hy, gy), \frac{d(kx, gy)+d(hy, fx)}{2}\}$ (1.4)

(iii) the pair (f, k) satisfies (CLR_f) property or the pair (g, h) satisfies (CLR_g) property,

(iv) the pairs (f, k) and (g, h) are weakly compatible.

Then the mappings f, g, h and k have a unique common fixed point.

The aim of this paper is to establish the best approximation results using the results of Verma and Pathak [9] and Kumar et. al.[5]. The results proved in this paper generalise and extend the various common fixed point results in complex valued metric spaces to the best approximation. We recall some definitions that will be used in our discussion:

2. Preliminaries:

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows: $z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Im(z_1) \leq Im(z_2)$.

It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$,

(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$,

(iii) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$,

(iv) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$

In particular, we will write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

(i) $0 \precsim z_1 \lnsim z_2 \implies |z_1| < |z_2|$;
(ii) $0 \precsim z_1 \precsim z_2 \implies |z_1| \leq |z_2|$;
(iii) $z_1 \precsim z_2$, $z_2 \prec z_3 \implies z_1 \prec z_3$;
(iv) $a, b \in \mathbb{R}$, $0 \leq a \leq b$ and $z_1 \precsim z_2 \implies az_1 \precsim bz_2$.

Definition 2.1: Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies:

1. $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

2. $d(x, y) = d(y, x)$ for all $x, y \in X$;

3. $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Remark 2.2: It is obvious that this concept is a generalization of the classical metric space. In fact, If d: X × X → ℝ satisfies above three conditions, then this *d* is a *metric* in the classical sense; that is, the following conditions are satisfied

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Example 2.3: Let $X = \mathbb{C}$. Define the mapping $d: X \times X \rightarrow \mathbb{C}$ by $d(z_1, z_2) = e^{ik} |z_1 - z_2|$, where $k \in \mathbb{R}$. Then (X, d) is a complex valued metric space.

Definition 2.4: The maximum function denoted by 'max' for the partial order relation $\preccurlyeq$ is defined by

(i) $\max \{z_1, z_2\} = z_2 \Leftrightarrow z_1 \preccurlyeq z_2$ or $|z_1| \leq |z_2|$

(ii) $z_1 \preccurlyeq \max \{z_2, z_3\} \Rightarrow z_1 \preccurlyeq z_2$ or $z_1 \preccurlyeq z_3$.

Remark 2.5: Let $z_1, z_2, z_3, \ldots \in \mathbb{C}$ and the partial order relation $\preccurlyeq$ is defined on $\mathbb{C}$. Then following statements are easy to prove

(i) If $z_1 \preccurlyeq \max \{z_2, z_3\}$ then $z_1 \preccurlyeq z_2$ if $z_3 \preccurlyeq z_2$;

(ii) If $z_1 \preccurlyeq \max \{z_2, z_3, z_4\}$ then $z_1 \preccurlyeq z_2$ if $\max \{z_3, z_4\} \preccurlyeq z_2$;

(iii) If $z_1 \preccurlyeq \max \{z_2, z_3, z_4, z_5\}$ then $z_1 \preccurlyeq z_2$ if $\max \{z_3, z_4, z_5\} \preccurlyeq z_2$, and so on.

Definition 2.6: Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is a natural number n_0 such that for all $n > n_0$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent sequence. We denote this by $\lim_n x_n = x$ or $x_n \rightarrow x$ as $n \rightarrow \infty$.

Definition 2.7: Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is a natural number n_0 such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called a Cauchy sequence in (X, d).

Definition 2.8: If every Cauchy sequence is convergent in X then (X, d) is called a complete complex valued metric space.

Definition 2.9: Let X be a non-empty set and $f: X \rightarrow X$ be a self map. Then $x \in X$ is a **fixed point** of f if $f(x) = x$. We denote by Fix (f), the set of all fixed points of f.

Definition 2.10: Let X be a metric space and $f, g: X \rightarrow X$. A point $x \in X$ is called

(1) a **coincidence point** of the pair (f, g) if $fx = gx$,

(2) a **common fixed point** of the pair (f, g) if x = fx = gx.

Definition 2.11: Let (X, d) be a metric space and M be a non empty subset of X. Let f, g: X → X. The pair {f, g} is said to be **weakly compatible** if they commute at their coincidence points, i.e., if fgx = gfx whenever fx = gx .

Example 2.12: Let X = ℂ. Define complex metric d: X ×X → ℂ by $d(z_1, z_2) = e^{ia}\,|z_1 - z_2|$, where a is any real constant. Then (X, d) is a complex valued metric space. Suppose f, g: X → X be defined as

$$fz = \begin{cases} 2e^{i\pi/4} & \text{if } Re(z) \neq 0, \\ 3e^{i\pi/3} & \text{if } Re(z) = 0. \end{cases} \quad \text{and } gz = \begin{cases} 2e^{i\pi/4}, & \text{if } Re(z) \neq 0, \\ 4e^{i\pi/6}, & \text{if } Re(z) = 0. \end{cases}$$

Then f and g are coincident when Re(z) ≠ 0 and $fz = gz = 2e^{i\pi/4}$. At this point $fgz = gfz = 2e^{i\pi/4}$. Hence the pair {f, g} commutes at their coincidence point. Therefore, it is weakly compatible at all z ∈ ℂ with Re(z) ≠ 0.

Definition 2.13: Let f, g: X → X be two self maps of a complex valued metric space (X, d). The pair {f, g} is said to satisfy property (E.A) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some t ∈ X.

Example 2.14: Let X = ℂ and d be any complex valued metric on X. Define f, g: X → X by $fz = \frac{1}{2}z^2$ and gz = − bz for all z ∈ X, where b is a fixed complex number and b ≠ 0. Consider a sequence $\{z_n\} = \{\frac{1}{n}\}$, n ≥ 1 in X, then $\lim_{n\to\infty} fz_n = 0$ and $\lim_{n\to\infty} gz_n = \lim_{n\to\infty} (\frac{-b}{n}) = 0$ as b ≠ 0. Hence, the pair {f, g} satisfies property (E.A) for the sequence $\{z_n\}$ in X with t = 0 ∈ X.

Remark 2.15: Pathak et al. has shown in [10] that weakly compatibility and property (E.A) are independent to each other.

Definition 2.16 [8]: Let f, g : X → X be two self maps of a complex valued metric space (X, d). The pair {f, g} is said to satisfy the (CLR_g)-Property, i.e., common limit in the range of g property if $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$ for some x ∈ X.

Example 2.17: Let X = ℂ and d be any complex valued metric on X. Define f, g : X → X by fz = z + 3i and gz = 4z for all z ∈ X. Consider a sequence $\{z_n\} = \{i + \frac{1}{n}\}$, n ≥ 1 in X, then $\lim_{n\to\infty} fz_n = \lim_{n\to\infty} z_n + 3i = \lim_{n\to\infty} (i + \frac{1}{n}) + 3i = 4i$, and $\lim_{n\to\infty} gz_n = \lim_{n\to\infty} 4(i + \frac{1}{n}) = 4i = g(0 + i)$. Hence the pair (f, g) satisfies (CLR_g)-Property in X with x = 0 + i ∈ X.

Definition 2.18: Let A be a subset of the set (ℂ, ≾). If there exists an element u of ℂ such that z ≾ u for all z in A, then A is bounded above and u is an upper bound. Similarly, if there exists l ∈ ℂ such that l ≾ z for all z in A, then A is bounded below and l is lower bound.

Definition 2.19: For a subset A ⊆ ℂ which is bounded above if there exists an upper bound s of A such that, for every upper bound u of A, s ≼ u, then the upper bound s is called supA. Similarly, for a subset A ⊆ ℂ which is bounded below if there exists a lower bound t of A such that for every lower bound l of A, l ≼ t, then the lower bound t is called inf A.

Remark 2.20: (i) Suppose that A⊆ ℂ is bounded above. Then there exists q = u + iv ∈ C such that z = x + iy ≼ q = u + iv, for all z ∈ A. It follows that x ≼ u and y ≼ v, for all z = x + iy ∈ A; that is, S = {x: z = x + iy ∈ A} and T = {y: z = x + iy ∈ A} are two sets of real numbers which are bounded above. Hence both sup S and Sup T exist. Let x^* = sup S and y^* = sup T. Then $z^* = x^* + iy^*$ is supA. Similarly, if A⊆ ℂ is bounded below, then $z' = x' + iy'$ is inf A, where x' = inf S = inf{x : z = x + iy ∈ A} and T = inf{y : z = x + iy ∈ A}

(ii) Any subset A⊆ ℂ which is bounded above has supremum. Equivalently, any subset A⊆ ℂ which is bounded below has infimum.

Definition 2.21: Let (X, d) be a complex valued metric space and M be any closed subset of X. If there exists a z_0 ∈ M such that d(z, z_0) = d(z, M) = $inf_{z_1 \epsilon M}\, d(z, z_1)$ then z_0 is called a **best approximation to z out of M**. We denote by $\boldsymbol{P_M}$**(z)**, the set of all best approximation to z out of M.

Main results:

Theorem 3.1:. Let (X, d) be a complex valued metric space and M be a subset of X. Let f, g, S, T: X→ X be four self-mappings, u be common fixed point of f, g, S, T and D = P_M(u). Suppose that

(i) the pairs (f, S) and (g, T) are weakly compatible and the pair (f, S) or (g, T) satisfies the property (E.A),

(ii) If SD = D, TD = D, f(∂M) ⊆ M, g(∂M) ⊆ M and D or fD or gD is complete,

(iii) for all x, y in D, (1.1) holds.

Then, f, g, S and T have a unique common fixed point in P_M(u).

Proof: Let y ∈ D then Sy ∈ D and Ty ∈ D. By the definition of P_M(u), y ∈ ∂M. Since f(∂M) ⊆ M and g(∂M) ⊆ M, it follows that fy, gy ∈ M. Now,

d(fy, u) = d(fy, gu)

≼ k max {d(Sy, Tu), d(gu, Sy), d(gu, Tu)} by (2.1),

= k max {d(Sy, u), d(u, Sy)},

= k d(Sy, u)

This implies d(fy, u) $\preccurlyeq$ d(Sy, u).

Hence fy ∈ M and Sy ∈ D implies that fy ∈ D. Similarly, gy ∈ D. Thus f, g, S and T are four self maps of D. Therefore by Theorem 1.1, there exists a unique z ∈ D such that z is common fixed point of f, g, S and T.

Theorem 3.2:. Let (X, d) be a complex valued metric space and M be a subset of X. Let f, g, h and k be four self-maps of X, u be common fixed point of f, g, h, k and D = P_M(u). Suppose that

(i) the pairs (f, k) and (g, h) are weakly compatible,

(ii) one of the pairs (f, k) or (g, h) satisfies the property (E.A.),

(iii) If hD = D, kD = D, f(∂M) ⊆ M, g(∂M) ⊆ M and k(P_M(u)) or h(P_M(u)) is closed,

(iv) for all x, y in D, (1.3) holds.

Then, f, g, h and k have a unique common fixed point in P_M(u).

Proof: Let y ∈ D then hy ∈ D and ky ∈ D. By the definition of P_M(u), y ∈ ∂M. Since f(∂M) ⊆ M and g(∂M) ⊆ M, it follows that fy, gy ∈ M. Now,

d(fy, u) = d(fy, gu)

$\preccurlyeq \lambda$ max {d(ky, hu), d(ky, fy), d(hu, gu), $\frac{d(ky,\ gu)+d(hu,\ fy)}{2}$} by (1.3)

= max {d(ky, u), d(ky, fy), $\frac{d(ky,u)+d(u,fy)}{2}$ }

$\preccurlyeq$ max {d(ky, u), [d(ky,u) + d(u, fy)], $\frac{d(ky,u)+d(u,fy)}{2}$ }

= max {d(fy, u), [d(ky, u) + d(u, fy)]}

$\preccurlyeq$ d(ky, u).

Thus d(fy, u) $\preccurlyeq$ d(ky, u).

Hence fy ∈ M and ky ∈ D implies that fy ∈ D. Similarly, gy ∈ D. Thus f, g, S and T are four self maps of D. Therefore by Theorem 1.3, there exists a unique z ∈ D such that z is common fixed point of f, g, h and k.

Example 3.3: Let X = [0, 3] be complex valued metric space with d(x, y) =| x - y |i and M = [1, 2]. Let f, g, h and k be self-maps of X defined by:

$$fx = \begin{cases} 0, & if\ x = 0 \\ \frac{1}{3}, & if\ x \in (0,1) \\ 1, & if\ x \in [1,3] \end{cases}$$

$$gx = \begin{cases} 0, & if\ x = 0 \\ \frac{1}{2}, & if\ x \in (0,1) \\ 1, & if\ x \in [1,3] \end{cases}$$

$$hx = \begin{cases} 0, & if\ x = 0 \\ \frac{1}{3}, & if\ x \in (0,1) \\ \frac{x+1}{2}, & if\ x \in [1,3] \end{cases}$$

$$kx = \begin{cases} 0, & if\ x = 0 \\ 1, & if\ x = 1 \\ \frac{1}{2}, & if\ x \in (0,1) \cup \{3\} \\ \frac{x-1}{2}, & if\ x \in (1,3) \end{cases}$$

Clearly, f(X) = {0, 1, $\frac{1}{3}$} ⊆ {0, $\frac{1}{3}$ } ∪ [1, 2] = h(X) and g(X) = {0, 1, $\frac{1}{2}$} ⊆ [0, 1] = k(X) and the pairs (f, k) and (g, h) are weakly compatible. Also, for the sequence x_n = {3 - $\frac{1}{n}$} in X $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} kx_n$ = 1 ∈ X and hence the pair (f, k) satisfies (E.A.) - property. Clearly, $P_M(0)$ = {1} and '1' is the unique common fixed point of f, g, h and k in P_M(u).

Theorem 3.4:. Let (X, d) be a complex valued metric space and M be a subset of X. Let f, g, S, T: X → X be four self-mappings, u be common fixed point of f, g, S, T and D = P_M(u). Suppose that

(i) the pairs (f, S) and (g, T) are weakly compatible
(ii) the pair (f, S) satisfy (CLR_f) property, or the pair (g, T) satisfy (CLR_g) property,
(iii) If SD = D, TD = D, f(∂M) ⊆ M, g(∂M) ⊆ M
(iv) for all x, y in D, (1.2) holds.

Then, f, g, S and T have a unique common fixed point in P_M(u).

Proof: Let y ∈ D then Sy ∈ D and Ty ∈ D. By the definition of P_M(u), y ∈ ∂M. Since f(∂M) ⊆ M and g(∂M) ⊆ M, it follows that fy, gy ∈ M. Now,

d(fy, u) = d(fy, gu)

≾ k max {d(Sy, Tu), d(gu, Sy), d(gu, Tu)} by (1.2),

= k max {d(Sy, u), d(u, Sy)},

$= k\, d(Sy, u)$

This implies $d(fy, u) \preccurlyeq d(Sy, u)$.

Hence fy ∈ M and Sy ∈ D implies that fy ∈ D. Similarly, gy ∈ D. Thus f, g, S and T are four self maps of D. Therefore by Theorem 1.2, there exists a unique z ∈ D such that z is common fixed point of f, g, S and T.

Theorem 3.5:. Let (X, d) be a complex valued metric space and M be a subset of X. Let f, g, h and k be four self-maps of X, u be common fixed point of f, g, h, k and D = $P_M(u)$. Suppose that

(i)the pairs (f, k) and (g, h) are weakly compatible,

(ii)the pair (f, k) satisfies (CLR_f) property or the pair (g, h) satisfies (CLR_g) property,

(iii)If hD = D, kD = D, f(∂M) ⊆ M, g(∂M) ⊆ M,

(iv) for all x, y in D, (1.4) holds.

Then, f, g, h and k have a unique common fixed point in $P_M(u)$.

Proof: Let y ∈ D then hy ∈ D and ky ∈ D. By the definition of $P_M(u)$, y ∈ ∂M. Since f(∂M) ⊆ M and g(∂M) ⊆ M, it follows that fy, gy ∈ M. Now,

$d(fy, u) = d(fy, gu)$

$\preccurlyeq \lambda \max \{d(ky, hu), d(ky, fy), d(hu, gu), \frac{d(ky, gu) + d(hu, fy)}{2}\}$ by (1.4)

$= \max \{d(ky, u), d(ky, fy), \frac{d(ky,u) + d(u,fy)}{2}\}$

$\preccurlyeq \max \{d(ky, u), [d(ky,u) + d(u, fy)], \frac{d(ky,u) + d(u,fy)}{2}\}$

$= \max \{d(fy, u), [d(ky, u) + d(u, fy)]\}$

$\preccurlyeq d(ky, u)$.

Thus $d(fy, u) \preccurlyeq d(ky, u)$.

Hence fy ∈ M and ky ∈ D implies that fy ∈ D. Similarly, gy ∈ D. Thus f, g, S and T are four self maps of D. Therefore by Theorem 1.4, there exists a unique z ∈ D such that z is common fixed point of f, g, h and k.

Example 3.6: Let X = [0, 3] be complex valued metric space with d(x, y) =| x - y |i and M = [1, 2]. Let f, g, h and k be self-maps of X defined by:

$$fx = \begin{cases} 0, & \text{if } x = 0 \\ \frac{2}{3}, & \text{if } x \in (0,1) \\ 1, & \text{if } x \in [1,3] \end{cases}$$

$$gx = \begin{cases} 0, & \text{if } x = 0 \\ \frac{1}{2}, & \text{if } x \in (0,1) \\ 1, & \text{if } x \in [1,3] \end{cases}$$

$$hx = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x = 1 \\ \frac{4}{3}, & \text{if } x \in (0,1) \\ \frac{x}{2}, & \text{if } x \in (1,3] \end{cases}$$

$$kx = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x = 1 \\ \frac{3}{2}, & \text{if } x \in (0,1) \\ \frac{x}{3}, & \text{if } x \in (1,3] \end{cases}$$

Clearly, $f(X) = \{0, 1, \frac{2}{3}\} \subseteq (\frac{1}{2}, \frac{3}{2}] = h(X)$ and $g(X) = \{0, 1, \frac{1}{2}\} \subseteq (\frac{1}{3}, 1] \cup \frac{3}{2} = k(X)$ and the pairs (f, k) and (g, h) are weakly compatible. Also, for the sequence $x_n = \{3 - \frac{1}{n}\}$ in X $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} kx_n = 1 \in f(X)$ and hence the pair (f, k) satisfies (CLR_f) - property. Clearly $P_M(0) = \{1\}$ and '1' is the unique common fixed point of f, g, h and k in $P_M(u)$.

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