Second Derivative Free Newton's Method

V.B. Kumar Vatti[1] Dept. of Engineering Mathematics,
Andhra University, Visakhapatnam, India,

Ramadevi Sri[2] Dept. of Engineering Mathematics,
Andhra University, Visakhapatnam, India,

M.S.Kumar Mylapalli[3] Dept. of Engineering Mathematics,
Gitam University, Visakhapatnam, India,

Abstract:
In this paper, we present a new two-step iterative method to solve the nonlinear equation $f(x)=0$ *and discuss about its convergence. Few numerical examples are considered to show the efficiency of the new method in comparison with the other methods considered in this paper.*

Keywords: *Chebyshev's method, Convergence, Iterative method, Newton's method, Nonlinear equation.*

1.INTRODUCTION

Many of the complex problems in Science and Engineering contains the function of nonlinear equation of the form

$$f(x)=0 \tag{1.1}$$

Where $f: I \to R$ for an open interval *I* is a scalar function.

Let x_{n+1} be the root of the equation (1.1) i.e., $f\left(x_{n+1}\right)=0$ while $f'\left(x_{n+1}\right)\neq 0$.

The classical quadratic convergent Newton's method **[3]** for finding the root of "(1.1)" is

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \tag{1.2}$$

$(n=0,1,2,\dots)$

The third order two step Chebyshev's method **[4]** is

$$\left.\begin{aligned} y_n &= x_n-\frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= y_n-\frac{(y_n-x_n)^2}{2}\cdot\frac{f''(x_n)}{f'(x_n)} \end{aligned}\right\} \tag{1.3}$$

$(n=0,1,2,\dots)$

The third order variant of Chebyshev's method **[9]** is

$$\left.\begin{aligned} y_n &= x_n-\frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= y_n-\frac{(y_n-x_n)}{2}\cdot\frac{f'(y_n)-f'(x_n)}{f'(x_n)} \end{aligned}\right\} \tag{1.4}$$

$(n=0,1,2,...)$

In this paper, we present a two step variant of Chebyshev's method in section 2. In section 3, the convergence criterion of the new method is discussed where as in the concluding section several numerical examples are considered to exhibit the efficiency of the developed method.

II. SECOND DERIVATIVE FREE NEWTON'S VARIANT METHOD

Following the basic assumption of Abbasbandy and Maheshweri**[1, 2]** and also others **[5, 7]**, we consider the second degree Taylor's expansion of $f\left(x_{n+1}\right)$ about x_n is

$$f\left(x_{n+1}\right)=f\left(x_n\right)+\left(x_{n+1}-x_n\right)f'\left(x_n\right)+\frac{\left(x_{n+1}-x_n\right)^2}{2}f''\left(x_n\right)$$

(2.1)

Where $x_{n+1}-x_n=h$

$$f\left(x_{n+1}\right)=x^2{}_{n+1}\left[\frac{f''\left(x_n\right)}{2}\right]+x_{n+1}\left[f'\left(x_n\right)-x_n f''\left(x_n\right)\right]+\left[f\left(x_n\right)-x_n f'\left(x_n\right)+\frac{x_n{}^2 f''\left(x_n\right)}{2}\right]$$

(2.2)

Since, x_{n+1} be the root of the equation "(1.1)" i.e., $f\left(x_{n+1}\right)=0$ then the "(2.2)" becomes

$$x^2{}_{n+1}\left[f''\left(x_n\right)\right]+x_{n+1}\left[2f'\left(x_n\right)-2x_n f''\left(x_n\right)\right]+\left[2f\left(x_n\right)-2x_n f'\left(x_n\right)+x_n{}^2 f''\left(x_n\right)\right]=0$$

(2.3)

The third order Newton's variant method **[8]** for finding the root of "(2.3)" is

$$x_{n+1}=x_n-\frac{f\left(x_n\right)}{f'\left(x_n\right)}\left[\frac{2}{1+\sqrt{1-2\omega_n}}\right] \qquad (2.4)$$

$(n=0,1,2,...)$

where, $\omega_n=\dfrac{f\left(x_n\right)f''\left(x_n\right)}{\left[f'\left(x_n\right)\right]^2}$.

Rewriting $f''\left(x_n\right)=\dfrac{f'\left(y_n\right)-f'\left(x_n\right)}{y_n-x_n}$ and $\omega_n=\dfrac{2f\left(y_n\right)}{f\left(x_n\right)}$ in "(2.4)" gives the two step Newton's variant method as

$$\left.\begin{aligned} y_n &= x_n-\frac{f\left(x_n\right)}{f'\left(x_n\right)} \\ x_{n+1} &= x_n-\frac{f\left(x_n\right)}{f'\left(x_n\right)}\cdot\frac{2}{\left[1+\sqrt{1-4\dfrac{f\left(y_n\right)}{f\left(x_n\right)}}\right]} \end{aligned}\right\} \qquad (2.5)$$

$(n=0,1,2,...)$

The simplified form of "(2.5)" can be rewritten as

$$x_{n+1}=x_n-\frac{f\left(x_n\right)}{f'\left(x_n\right)}\cdot\frac{1}{2\omega_n}\left[1-\left(1-4\omega_n\right)^{1/2}\right]$$

Expanding $(1-4\omega_n)^{1/2}$ up to three terms i.e., up to ω_n^2, we get the required two step Newton's variant method as

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)}\left[1+\frac{f(y_n)}{f(x_n)}\right] \end{aligned}\right\} \quad (2.6)$$

$(n = 0,1,2,\ldots)$

III. CONVERGENCE CRITERIA

Theorem 3.1. Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \to R$ for an open interval *I*. Then, the new method that is defined by "(2.2)" has the third order convergence and satisfies the following error equation,

$$\varepsilon_{n+1} = 2c_2^2\varepsilon_n^3 + o\left(\varepsilon_n^4\right)$$

Where, $x_{n+1} = \varepsilon_{n+1} + \alpha$ and $x_n = \varepsilon_n + \alpha$

Proof: Let α be a simple zero of equation "(1.1)". By the Taylor's expansions

$$f(x_n) = f'(\alpha)\left[\varepsilon_n + c_2\varepsilon_n^2 + c_3\varepsilon_n^3 + o\left(\varepsilon_n^4\right)\right] \quad (3.1)$$

and $f'(x_n) = f'(\alpha)\left[1+2c_2\varepsilon_n + 3c_3\varepsilon_n^2 + 4c_4\varepsilon_n^3 + o\left(\varepsilon_n^4\right)\right]$ (3.2)

Dividing "(3.2)" by "(3.1)", we have

$$\left[\frac{f(x_n)}{f'(x_n)}\right] = \left[\varepsilon_n - (c_2)\varepsilon_n^2 + 2\left(c_2^2 - c_3\right)\varepsilon_n^3 + 0\left(\varepsilon_n^4\right)\right] \quad (3.3)$$

$$f(y_n) = f'(\alpha)\left[c_2\varepsilon_n^2 + 2\left(c_3 - c_2^2\right)\varepsilon_n^3 + o\left(\varepsilon_n^4\right)\right] \quad (3.4)$$

Dividing "(3.4)" by "(3.1)", we obtain

$$\frac{f(y_n)}{f(x_n)} = c_2\varepsilon_n - \left(2c_3 - 3c_2^2\right)\varepsilon_n^2 + \left(-10c_2c_3 + 3c_4 + 9c_2^3\right)\varepsilon_n^3 + o\left(\varepsilon_n^4\right)$$

(3.5)

Adding '1' on both sides to "(3.5)", we have

$$1+\frac{f(y_n)}{f(x_n)} = 1 + c_2\varepsilon_n - \left(2c_3 - 3c_2^2\right)\varepsilon_n^2 + \left(-10c_2c_3 + 3c_4 + 9c_2^3\right)\varepsilon_n^3 + o\left(\varepsilon_n^4\right)$$

(3.6)

Multiplying "(3.3)" and "(3.6)", we get

$$\frac{f(x_n)}{f'(x_n)}\left[1+\frac{f(y_n)}{f(x_n)}\right] = \varepsilon_n + c_2\varepsilon_n^2 + \left(2c_3 - 3c_2^2\right)\varepsilon_n^3 - c_2\varepsilon_n^2 + c_2^2\varepsilon_n^3 + 2\left(c_2^2 - c_3\right)\varepsilon_n^3 + o\left(\varepsilon_n^4\right)$$

$$= \varepsilon_n - 2c_2^2\varepsilon_n^3 + o\left(\varepsilon_n^4\right) \quad (3.7)$$

Thus, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}\left[1+\dfrac{f(y_n)}{f(x_n)}\right]$ becomes

$$\varepsilon_{n+1}+\alpha=(\varepsilon_n+\alpha)-\varepsilon_n+2c_2^{\,2}\varepsilon_n^{\,3}+o\left(\varepsilon_n^{\,4}\right)$$

$$\varepsilon_{n+1}=2c_2^{\,2}\varepsilon_n^{\,3}+o\left(\varepsilon_n^{\,4}\right) \tag{3.8}$$

Equation "(3.8)" establishes the third order convergence of the method that is defined by "(2.6)".

IV. NUMERICAL EXAMPLES

We consider few numerical examples considered by **[6, 7]** and the method "(2.6)" is compared with themethods "(1.2)", "(1.3)" and "(1.4)". The computational results are tabulated below and the results are correct up to an error less than ε as indicated for each of the problems.

Example 1. Consider the following equation, $f(x)=e^x-3x^2-=0$.

TABLE 1.

The results obtained by three methods for solving $f(x)=e^x-3x^2-=0$ with $x_0=0.5$ and ε=0.5E–20

Formula	No. of iterations (n)	Root x_n
Newton	7	0.91000757248870907904
Chebyshev	---	DIVERGENT
A Variant of Chebyshev	7	0.91000757248870907904
New Method	6	0.91000757248870907904

Example 2. Consider the following equation, $f(x)=x^3-e^{-x}=0$

TABLE 2.

The results obtained by four methods for solving $f(x)=x^3-e^{-x}=0$ with $x_0=-1$ and $\varepsilon=0.5E-20$

Formula	No. of iterations (n)	Root x_n
Newton	8	0.77288295914921017344
Chebyshev	7	0.77288295914921017344
A Variant of Chebyshev	7	0.77288295914921017344
New Method	6	0.77288295914921017344

Example 3 . Consider the following equation, $f(x) = \sin x - 0.5x = 0$

TABLE 3.

The results obtained by four methods for solving $f(x) = \sin x - 0.5x = 0$ with $x_0 = 3$ and $\varepsilon = 0.5E-20$

Formula	No. of iterations (n)	Root x_n
Newton	6	1.89549426703398109184
Chebyshev	5	1.89549426703398109184
A Variant of Chebyshev	5	1.89549426703398109184
New Method	4	1.89549426703398109184

Example 4. Consider the following equation, $f(x) = x^3 - 2x - 5 = 0$

TABLE 4.

The results obtained by four methods for solving $f(x) = x^3 - 2x - 5 = 0$ with $x_0 = 3$ and $\varepsilon = 0.5E-20$

Formula	No. of iterations (n)	Root x_n
Newton	7	2.09455148154232668160
Chebyshev	5	2.09455148154232668160
A Variant of Chebyshev	5	2.09455148154232668160
New Method	4	2.09455148154232668160

Example 5. Consider the following equation, $f(x) = \sin x = 0$

TABLE 5.

The results obtained by four methods for solving $f(x) = \sin x = 0$ with $x_0 = 0.5$ and $\varepsilon = 0.5E-20$

Formula	No. of iterations (n)	Root x_n
Newton	5	0
Chebyshev	4	0
A Variant of Chebyshev	4	0
New Method	3	0

V. CONCLUSION

With the number of iterations tabulated for each of the methods for five non-linear equations, we conclude that the method (2.6) is efficient one compared to the methods considered in this paper.

REFERENCES

[1] Abbasbandy, S. Improving Newton-Raphson method for nonlinear equations by modified Adomian decomposition method, *Applied Mathematics and Computation*, Vol. 145, 2003, pp. 887 – 893.

[2] AmitkumarMaheshwari, A fourth order iterative method for solving nonlinear eqations, *Applied Mathematics and computation*, Vol.211, 2009, pp. 383-391.

[3] AvramSidi, Unified treatment of regular falai, Newton–Raphson, Secent, and Steffensen methods for nonlinear equations, *Journal of Online Mathematics and Its Applications*. 2006, pp. 1-13.

[4] Gutierrez, J.M., Hernandez, M.A., Afamilyof Chebyshev-Halley type methods in Banach spaces. *Bull. Aust. Math. Soc.* 55, 113-130(1997).

[5] Jinhai Chen, Weiguo Li, On new exponential quadratically convergent iterative formulae,, *Applied Mathematics and Computation*,Vol. 180, 2006, pp. 242-246.

[6] Nasr Al-Din Ide, A new Hybrid iteration method for solving algebraic equations, *Applied Mathematics and Computation*,Vol. 195, 2008, pp. 772-774.

[7] Xing-GuoLuo, A note on the new iterative method for solving algebraic equation, *Applied Mathematics and Computation*, Vol. 171, No 2, 2005, pp. 1177- 1183.

[8] VattiV.B.Kumar., Ramadevi Sri., Mylapalli M. S. Kumar., A Newton's Variant third order method, *Engineering Science and Technology: An International Journal*, Vol. 6, No. 4, 2016.

[9] VattiV.B.Kumar., Ramadevi Sri., Mylapalli M. S. Kumar., A variant of Chebyshev's method, *Bulletin of Pure and Applied Sciences Mathematics (E),* 2017.

AUTHOR'S PROFILE

1. V.B. Kumar, Vatti received the Doctoral of Philosophy in Mathematics from Indian Institute of Technology, Mumbai in 1987. Presently he is working as a Professor of Engineering Mathematics in Andhra University. He has nearly above 30 years of teaching experience and his research interest includes developing application oriented numerical methods.

2. Ramadevi Sri received the Master's degree in Mathematics from Andhra University, Visakhapatnam in 2009. Currently, she is pursuing for Ph.D. degree in Engineering Mathematics in Andhra University. Her research interest includes developing numerical methods for solving non-linear equations.

3. M.S. Kumar, Mylapalli received the Doctoral of Philosophy in Applied Mathematics from Andhra University, Visakhapatnam in 2010. Presently he is working as a Assistant Professor of Mathematics in Gitam University. He has nearly above 11 years of teaching experience and his research interest includes developing numerical methods for solving nonlinear equation, system of linear and nonlinear equations.