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ON SOME FRACTIONAL DERIVATIVES

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Abstract:
In the present paper, the authors approach is based on the use of fractional derivatives in relationship with multivariable H-function and others functions, which will be useful to analysis the various problems in different fields. And some special cases are discussed at the lost section of the paper.

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Introduction:

The fractional derivatives is an extension of the ordinary calculus[1],has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering[2].

Samko. et. Al. [4] has compiled differential aspects of fractional derivatives and integration in their text book. Ali and Kalla [5] have dealt with application of fractional calculus to the solution of general terms of differential equations. Monreal. et. Al. [6] have worked for special functions of one and more variables is important, such as in the evaluation of series and integrals, the derivation of generation functions and the solution of differential and integral equations.

Definition and preliminaries used

The H-function of several complex variables, defined H. M. Srivastava and R. Panda [1], we will define and represent it in the following from [1, page 252, equation (C.1)]

$$H[z_1,\dots,z_r] = H^{0,N:\,M^{(1)},N^{(1)};\dots;M^{(r)},N^{(r)}}_{P,Q:\,P^{(1)},Q^{(1)};\dots;P^{(r)},Q^{(r)}}$$

$$\times\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left|\begin{matrix} (a_j,\alpha_j^{(1)},\dots,\alpha_j^{(r)})_{1,P}\colon \left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,P^{(1)}};\dots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,P^{(r)}} \\ (b_j,\beta_j^{(1)},\dots,\beta_j^{(r)})_{1,Q}\colon \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,Q^{(1)}};\dots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,Q^{(r)}} \end{matrix}\right.\right]$$

$$= \frac{1}{(2\pi i)^r}\int_{L_1}\dots\int_{L_r}\phi_1(\xi_1)\dots\phi_r(\xi_r)\;\psi(\xi_1,\dots,\xi_r)\,{z_1}^{\xi_1},\dots,{z_r}^{\xi_r}\,d\xi_1\dots d\xi_r \qquad (1)$$

where $i=\sqrt{-1}$,

$$\phi_k\xi_k = \frac{\prod_{j=1}^{M^{(k)}}\Gamma(d_j^{(k)}-\delta_j^{(k)}\xi_k)\prod_{j=1}^{M^{(k)}}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}\xi_k)}{\prod_{j=M^{(k)}+1}^{Q^{(k)}}\Gamma(1-d_j^{(k)}+\delta_j^{(k)}\xi_{(k)})\prod_{j=N^{(k)}+1}^{P^{(k)}}\Gamma(c_j^{(k)}-\gamma_j^{(k)}\xi_k)} \qquad (2)$$

for all $k \in \{1, \ldots, r\}$ and

$$\psi(\xi_1, \ldots, \xi_r) = \frac{\prod_{j=1}^{N} \Gamma(1-a_j+\sum_{k=1}^{r} \alpha_j^{(k)} \xi_k)}{\prod_{j=N+1}^{P} \Gamma(a_j-\sum_{k=1}^{r} \alpha_j^{(k)} \xi_k) \prod_{j=1}^{Q} \Gamma(1- b_j+\sum_{k=1}^{r} \beta_j^{(k)} \xi_k)} . \quad (3)$$

Here, for convenience, $(a_j, \alpha_j^{(1)}, \ldots, \alpha_j^{(r)})_{1,P}$ abbreviates the p-number array

$$((a_1, \alpha_1^{(1)}, \ldots, \alpha_1^{(r)})(a_p, \alpha_p^{(1)}, \ldots, \alpha_p^{(r)})) \quad (4)$$

while $\left(, \gamma_j^{(k)}\right)_{1,P^{(k)}}$ abbreviates the array of $(P)^k$ pairs of parameters:

$$\left(c_1^{(k)}, \gamma_1^{(k)}\right); \ldots ; \left(c_{(P)^k}^{(k)}, \gamma_{(P)^k}^{(k)}\right) \qquad (k = 1, \ldots, r) \quad (5)$$

and so on suppose, as usual, that the parameters:

$$\begin{matrix} a_j, \; j=1,\ldots,P; & c_j^{(k)}, \; j=1,\ldots,P^{(k)} \\ b_j, \; j=1,\ldots,Q; & d_j^{(k)}, \; j=1,\ldots,Q^{(k)} \end{matrix} \qquad (\forall\, k \in \{1, \ldots, r\} \quad (6)$$

are complex numbers and the associated coefficients

$$\begin{matrix} \alpha_j^{(k)}, \; j=1,\ldots,P; & \gamma_j^{(k)}, \; j=1,\ldots,P^{(k)} \\ \beta_j^{(k)}, \; j=1,\ldots,Q; & \delta_j^{(k)}, \; j=1,\ldots,Q^{(k)} \end{matrix} \qquad (\forall\, k \in \{1, \ldots, r\} \quad (7)$$

are positive real numbers such that

$$\Lambda_k := \sum_{j=1}^{P} \alpha_j^{(k)} - \sum_{j=n}^{Q} \beta_j^{(k)} + \sum_{j=1}^{P^{(k)}} \gamma_j^{(k)} - \sum_{j=1}^{Q^{(k)}} \delta_j^{(k)} \leq 0, \quad (8)$$

$$\Omega_k := -\sum_{j=N+1}^{P} \alpha_j^{(k)} - \sum_{j=n}^{Q} \beta_j^{(k)} + \sum_{j=1}^{N^{(k)}} \gamma_j^{(k)} - \sum_{j=N^{(k)}+1}^{P^{(k)}} \gamma_j^{(k)}$$

$$+ \sum_{j=1}^{M^{(k)}} \delta_j^{(k)} - \sum_{j=M^{(k)}+1}^{Q^{(k)}} \delta_j^{(k)} > 0, \qquad \forall\, k \in \{1, \ldots, r\} \quad (9)$$

where the integers $N, P, Q, M^{(k)}, N^{(k)}, P^{(k)} and\ Q^{(k)}$ are constrained by the inequalities $0 \le N \le P, Q \ge 0, 1 \le M^{(k)} \le Q^{(k)}\ and\ 0 \le N^{(k)} \le P^{(k)}$ (for all $k \in \{1, \dots, r\}$ and the equality in (8) holds true for suitably restricted values of the complex variables $z_1, \dots, z_r$.

The multiple Melin-Barnes contour integral [3, page 251, equation (C.1)] representing the multivariable H-function (1) converges absolutely, under the conditions (9), when

$$|\arg(z_k)| < \frac{1}{2}\Omega_k \pi \qquad (\forall\, k \in \{1, \dots, r\}) \tag{10}$$

the points $z_k = 0\ (k = 1, \dots, r)$ and various exceptional parameter values being tacitly excluded. Furthermore, we have (cf. [9, page 131, equation 1.9]):

$$H[z_1, \dots, z_r] = \begin{cases} O\left(|z_1|^{\xi_1} \dots |z_r|^{\xi_r}\right), & (\max\{|z_1|, \dots, |z_r|\} \to 0), \\ O(|z_1|^{\eta_1} \dots |z_r|^{\eta_r}), & (N = 0; \max\{|z_1|, \dots, |z_r|\} \to \infty), \end{cases} \tag{11}$$

where $(k = 1, \dots, r)$

$$\xi_k = \min\left\{\frac{Re\ (d_j^{(k)})}{\delta_j^{(k)}}\right\}, \qquad \left(j = 1, \dots, M^{(k)}\right), \tag{12}$$

$$\eta_k = \min\left\{\frac{Re\ (c_j^{(k)} - 1)}{\gamma_j^{(k)}}\right\}, \qquad \left(j = 1, \dots, N^{(k)}\right), \tag{13}$$

provided that each of the inequalities in (8)-(10) holds true.

Fractional Derivatives:

The notation that is used to denote the fractional derivative is $D^\alpha f(x)$ for any arbitrary number of order α. Fractional derivative can be defined in terms of the fractional integral [2] as follows

$$D^\alpha f(x) = D^n[D^{-n} \mathrm{f(t)}] \tag{14}$$

where $0 < u < 1$ and n is the smallest integer greater than α such that $u = n - \alpha$.

$$D^\alpha\left(f(t)\right) = D^m\, I^{m-\alpha}\, f(t)$$

$$D^\alpha f(t) = \begin{Bmatrix} \frac{dm}{dt^m}\left[\frac{1}{\Gamma(m-\alpha}\int_0^t \frac{f(\lambda)}{(t-\lambda)} d\lambda\right] m - 1 < \alpha < m \\ \frac{dm}{dt^m}\ f(t) \qquad \alpha = m \end{Bmatrix} \tag{15}$$

2. Main Results:

In this section, we evaluate following some fractional derivatives involving multivariable H-function:

Theorem 1:

$$D_x^{\mu}\{X^k(X^{\nu_1}+a)^{\lambda}(b-X^{\nu_2})^{-\delta} \quad H[Z_1X^{-\rho_1}(X^{\nu_1}+a)^{\sigma_1}(b-X^{\nu_2})^{-\delta_1},\ldots,$$
$$Z_rX^{-\rho_r}(X^{\nu_1}+a)^{\sigma_r}(b-X^{\nu_2})^{-\delta_r}]\}$$

$$= a^{\lambda}b^{-\delta}X^{k-\mu}\sum_{m,n=0}^{\infty}\frac{\left(\frac{X^{\nu_1}}{a}\right)^m\left(\frac{X^{\nu_1}}{b}\right)^n}{m!n!} \times H^{0,N+2:\,M^{(1)},N^{(1)};\ldots;M^{(r)},N^{(r)}}_{P+3,Q+3:\,P^{(1)},Q^{(1)};\ldots;P^{(r)},Q^{(r)}}$$

$$\times\left[\begin{matrix}Z_1X^{-\rho_1}a^{\sigma_1}b^{-\delta_1}\\ \vdots\\ Z_rX^{-\rho_r}a^{\sigma_r}b^{-\delta_r}\end{matrix}\left|\begin{matrix}(-\lambda;\sigma_1,\ldots,\sigma_r),(1-\delta-n;\delta_1,\ldots,\delta_r),(a_j,\alpha_j^{(1)},\ldots,\alpha_j^{(r)})_{1,P}\\ (1+k+\nu_1m+\nu_2n;\rho_1,\ldots,\rho_r),(b_j,\beta_j^{(1)},\ldots,\beta_j^{(r)})_{1,Q}\end{matrix}\right.\right.$$
$$\left.\begin{matrix}(1+k-\mu+\nu_1m+\nu_2n;\rho_1,\ldots,\rho_r):\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,P^{(1)}};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,P^{(r)}}\\ (-\lambda+m;\sigma_1,\ldots,\sigma_r),(1-\delta;\delta_1,\ldots,\delta_r)\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,Q^{(1)}};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,Q^{(r)}}\end{matrix}\right]$$

(16)

provided (in addition to the usual convergence and existence conditions) that $\min\{\nu_1,\nu_2,\rho_i,\sigma_i,\delta_i\}>0$, $i=1,\ldots,r;$ $max\left\{\left|\arg\left(\frac{X}{a}\right)\right|,\left|\arg\left(\frac{X}{b}\right)\right|\right\}<\pi, Re(\mu)<\pi$ and

$$Re(\kappa)+\sum_{i=1}^{r}\rho_i a_i b_i > -1, \tag{17}$$

where $a_1,\ldots,a_r;b_1,\ldots,b_r$ are given by (12).

Theorem2:

$${}_XD_{\infty}^{\mu}\{X^k(X^{\nu_1}+a)^{\lambda}(b-X^{\nu_2})^{-\delta}H[Z_1X^{\rho_1}(X^{\nu_1}+a)^{-\sigma_1}(b-X^{\nu_2})^{-\delta_1},\ldots,$$
$$Z_rX^{\rho_r}(X^{\nu_1}+a)^{-\sigma_r}(b-X^{\nu_2})^{-\delta_r}]\}$$

$$= a^{\lambda}b^{-\delta}X^{k-\mu}(-1)^{\mu}\sum_{m,n=0}^{\infty}\frac{\left(\frac{X^{\nu_1}}{a}\right)^m\left(\frac{X^{\nu_1}}{b}\right)^n}{m!n!} \times H^{0,N+1:\,M^{(1)},N^{(1)};\ldots;M^{(r)},N^{(r)}}_{P+3,Q+3:\,P^{(1)},Q^{(1)};\ldots;P^{(r)},Q^{(r)}}$$

$$\times \left[\begin{matrix} z_1 X^{\rho_1} a^{-\sigma_1} b^{-\delta_1} \\ \vdots \\ z_r X^{\rho_r} a^{-\sigma_r} b^{-\delta_r} \end{matrix}\middle| \begin{matrix} (1-\delta-n;\delta_1,\dots,\delta_r),(a_j,\alpha_j^{(1)},\dots,\alpha_j^{(r)})_{1,P},(1+\lambda-m;\sigma_1,\dots,\sigma_r), \\ (1+\lambda;\sigma_1,\dots,\sigma_r),(b_j,\beta_j^{(1)},\dots,\beta_j^{(r)})_{1,Q},(-k+\mu-mv_1-nv_2;\rho_1,\dots,\rho_r) \end{matrix}\right.$$

$$\left.\begin{matrix} (-k-v_1 m-v_2 n;\rho_1,\dots,\rho_r):\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,P^{(1)}};\dots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,P^{(r)}} \\ (1-\delta;\delta_1,\dots,\delta_r):\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,Q^{(1)}};\dots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,Q^{(r)}} \end{matrix}\right] \tag{18}$$

provided (in addition to the usual convergence and existence conditions) that $\min\{v_1, v_2, \rho_i, \sigma_i, \delta_i\} > 0$, $i = 1, \dots, r;\ max\left\{\left|\arg\left(\frac{X}{a}\right)\right|, \left|\arg\left(\frac{X}{b}\right)\right|\right\} < \pi, Re(\mu) < \pi$ and

$$Re(\kappa) + \sum_{i=1}^{r} \rho_i a_i b_i > -1, \tag{19}$$

Where $a_1, \dots, a_r; b_1, \dots, b_r$ are given by (12).

Proof:

To prove of (16), we first replace the multivariable H-function occurring on the L.H.S. by its Mellin-Barnes contour integrals, collected the powers of $X, (X^{v_1} + a)$ and $(b - X^{v_2})$ and apply binomial expansion:

$$(X+\xi)^\lambda = \xi^\lambda \sum_{s=0}^{\infty} \binom{\lambda}{s} \left(\frac{X}{\xi}\right)^s;\ \left|\frac{X}{\xi}\right| < 1 \tag{20}$$

we then apply the formula [3, page 67 equation. (4.4.4)]:

$$D_X^\mu = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda-\mu)} X^{\lambda-\mu}; \quad (Re(\lambda) > -1), \tag{21}$$

and interpret the resulting Mellin-Barnes contour integrals as a multivariable H-function, we shall arrive at (16)

On similar manner as given above and using formula Shrivastava [1, page 21, equation (2.7.8)]

$${}_X D_\infty^\mu Z^{-\lambda} = \frac{(-1)^q \Gamma(\lambda+q)}{\Gamma(\lambda)} Z^{-\lambda-q}, \text{ q arbitrary,} \tag{22}$$

we can easily derive the result from (18)

Special Cases:

Put $z_1 = z_2 = z_3 = \cdots z_r = z$, $\alpha_j^{(1)} = \alpha_j^{(2)} = \alpha_j^{(3)} = \cdots \alpha_j^{(r)} = \alpha_j$, $\beta_j^{(1)} = \beta_j^{(2)} = \beta_j^{(3)} = \cdots \beta_j^{(r)} = \beta_j$, $\left(c_j^{(1)}, \gamma_j^{(1)}\right) = \left(c_j^{(2)}, \gamma_j^{(2)}\right) = \left(c_j^{(3)}, \gamma_j^{(3)}\right) = \cdots \left(c_j^{(r)}, \gamma_j^{(r)}\right) = 0 = \left(d_j^{(1)}, \delta_j^{(1)}\right) = \left(d_j^{(2)}, \delta_j^{(2)}\right) = \left(d_j^{(3)}, \delta_j^{(3)}\right) = \cdots \left(d_j^{(r)}, \delta_j^{(r)}\right)$ and r = 1 (say) in theorem 1 and 2. We get a well known result as reported in [7]

(i) $D_x^{\mu}\{X^k(X^{\nu_1}+a)^{\lambda}(b-X^{\nu_2})^{-\delta} \quad H[ZX^{-\rho_1}(X^{\nu_1}+a)^{\sigma_1}(b-X^{\nu_2})^{-\delta_1}]\}$

$$= a^{\lambda}b^{-\delta}X^{k-\mu}(-1)^{\mu}\sum_{m,n=0}^{\infty}\frac{\left(\frac{X^{\nu_1}}{a}\right)^m\left(\frac{X^{\nu_1}}{b}\right)^n}{m!n!} \times H_{P+3,Q+3:}^{m+1,N+2:}$$

$$\times \left[Z_1 X^{-\rho_1}a^{\sigma_1}b^{-\delta_1}\middle|\begin{matrix}(-\lambda;\sigma_1),(1-\delta-n;\delta_1),(a_j,\alpha_j)_{1,P}\ (1+k-\mu+\nu_1 m+\nu_2 n;\rho_1,)\\(1+k+\nu_1 m+\nu_2 n;\rho_1),(b_j,\beta_j)_{1,Q}\quad(-\lambda+m;\sigma_1),(1-\delta;\delta_1);\end{matrix}\right]$$

(23)

(ii) ${}_XD_{\infty}^{\mu}\{X^k(X^{\nu_1}+a)^{\lambda}(b-X^{\nu_2})^{-\delta}H[Z_1X^{\rho_1}(X^{\nu_1}+a)^{-\sigma_1}(b-X^{\nu_2})^{-\delta_1}]\}$

$$= a^{\lambda}b^{-\delta}X^{k-\mu}(-1)^{\mu}\sum_{m,n=0}^{\infty}\frac{\left(\frac{X^{\nu_1}}{a}\right)^m\left(\frac{X^{\nu_1}}{b}\right)^n}{m!n!} \times H_{P+3,Q+3}^{m+2,N+1}\Big[Z_1X^{\rho_1}a^{-\sigma_1}b^{-\delta_1}\Big|$$

$$\left.\begin{matrix}(1-\delta-n;\delta_1),(a_j,\alpha_j)_{1,P},(1+\lambda-m;\sigma_1),\quad(-k-\nu_1 m-\nu_2 n;\rho_1,\dots,\rho_r)\\(1+\lambda;\sigma_1),(b_j,\beta_j)_{1,Q},(-k+\mu-m\nu_1-n\nu_2;\rho_1)\quad(1-\delta;\delta_1)\end{matrix}\right]$$

(24)

Conclusion:

Here we conclude with the remark that the results and the operators proved in this paper appear to be new and likely to have useful applications to a wide range of problems of mathematics, statistics and physical sciences.

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