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Best Approximation Results via Common Fixed Points inComplex Valued Metric Spaces

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Abstract:

In this paper we introduce the concept of best approximation in the setting of complex valued metric spaces. By using this concept, we prove certain best approximation results whichextend and generalize various known results of ordinary metric spaces. We also give some suitable examples in supports of the proved results.

Keywords: Complex valued metric space; Common fixed point; Weakly Compatible maps; Best approximation.

Mathematics Subject Classification: 54H25, 47H10.

1. Introduction:

Since the appearance of the Banach contraction mapping principle, a number of articles have been dedicated to the improvement and generalization of that result. Most of these deal with the generalizations of the contractive condition in metric spaces. Many authors generalized and extended the notion of a metric spaces such as 2- metric spaces of Gahler [4], vector-valued metric spaces of Perov [7], G-metric spaces of Mustafa and Sims [6], cone metric spaces of Huang and Zhang [5], modular metric spaces of Chistyakov [3] etc.

In 2011, Azam et al. [1] introduced the complex valued metric space which is more general than well-known metric spaces and gave common fixed point theorems for mappings satisfying generalized contraction condition. They gave the following common fixed point result:

Theorem 1.1: Let (X, d) be a complex valued metric space, S and T be two self-maps of X satisfying

$$d(Tx,Ty) \precsim \lambda d(x, y) + \frac{\mu d(x,Tx)d(y,Ty)}{1+d(x,y)} \tag{1.1}$$

for all x,y ∈ X , where λ,μ are nonnegative reals with λ+μ< 1. Then S and T have a unique common fixed point.

Afterwards, Bhatt, Chaukiyal and Dimri [2] proved the common fixed point results for weakly compatible maps in complex valued metric space:

Theorem 1.2: Let (X, d) be a complex valued metric space and f, g, S and T be four self-maps of X such that T(X) ⊆f (X), S(X) ⊆g(X) and satisfying

$$d(Sx, Ty) \precsim ad(fx, gy) + b [d(fx, Sx) + d(gy, Ty)] + c [d(fx, Ty) + d(gy, Sx)] \tag{1.2}$$

for all x, y∈ X, where a, b, c ≥0 and a + 2b + 2c <1. Suppose that the pairs {f, S} and {g, T} are weakly compatible. Then f, g, S and T have a unique common fixed point.

In 2012, Tiwari and Shukla [2] proved the common fixed point theorems for six self maps having commuting and weakly compatible:

Theorem 1.3: Let (X, d) be a complex valued metric space and F, G, I, J, K and L be self maps of X satisfying the following conditions:

(i) KL(X) ⊆F (X) and IJ(X) ⊆ G(X),

(ii) d(IJx, KLy) ≾ad(Fx, Gy) + b [d(Fx, IJx) + d(Gy, KLy)] + c [d(Fx, KLy) + d(Gy, IJx)] for all x, y∈ X where a, b,c ≥0 and a + 2b + 2c <1. (1.3)

Suppose that the pairs {F, IJ} and {G, KL} are weakly compatible and the pairs {K, L}, {K, G}, {L, G}, {I, J}, {I, F} and {J, F}are commuting. Then F, G, I, J, K and L have a unique common fixed point in X.

Afterwards, Sastry et al. [8] generalised the results of Bhatt et. al. [2] by using the more general contractive condition:

Theorem 1.4: Let (X, d) be a complete complex valued metric space and f, g, S and T be four self-maps of X such that T(X) ⊆f (X), S(X) ⊆g(X) and satisfyfor all x, y∈ X,

$d(Sx, Ty) \precsim \lambda \max \{d(fx, gy), \frac{d(fx,Sx) + d(gy,Ty)}{2}, \frac{d(fx,Ty) + d(gy,Sx)}{2}\}$ (1.4)

$\lambda < 1$. Suppose that the pairs {f, S} and {g, T} are weakly compatible and T(X) is closed. Then f, g, S and T have a unique common fixed point.

Since then a number of common fixed point theorems have been established in complex valued metric spaces. In this paper, we use these results toestablish certain best approximation results in complex valued metric space which extend and improve various known results. We first give some preliminaries:

2. Preliminaries:

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows: $z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Im(z_1) \leq Im(z_2)$.

It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$,

(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$,

(iii) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$,

(iv) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$

In particular, we will write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

(i) $0 \precsim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|$;

(ii) $0 \precsim z_1 \precsim z_2 \Rightarrow |z_1| \leq |z_2|$;

(iii) $z_1 \precsim z_2$, $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$;

(iv) $a, b \in \mathbb{R}$, $0 \leq a \leq b$ and $z_1 \precsim z_2 \Rightarrow az_1 \precsim bz_2$.

Definition 2.1: Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies:

1. $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

2. $d(x, y) = d(y, x)$ for all $x, y \in X$;

3. $d(x, y) \preccurlyeq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Remark 2.2: It is obvious that this concept is a generalization of the classical metric space. In fact, If $d: X \times X \rightarrow \mathbb{R}$ satisfies above three conditions, then this *d* is a *metric* in the classical sense; that is, the following conditions are satisfied:

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Example 2.3: Let $X = \mathbb{C}$. Define the mapping $d: X \times X \rightarrow \mathbb{C}$ by $d(z_1, z_2) = e^{ik} |z_1 - z_2|$, where $k \in \mathbb{R}$. Then (X, d) is a complex valued metric space.

Definition 2.4: The maximum function denoted by 'max' for the partial order relation $\preccurlyeq$ is defined by

(i) $\max \{z_1, z_2\} = z_2 \Leftrightarrow z_1 \preccurlyeq z_2$ or $|z_1| \leq |z_2|$

(ii) $z_1 \preccurlyeq \max \{z_2, z_3\} \Rightarrow z_1 \preccurlyeq z_2$ or $z_1 \preccurlyeq z_3$.

Remark 2.5: Let $z_1, z_2, z_3, \ldots \in \mathbb{C}$ and the partial order relation $\preccurlyeq$ is defined on $\mathbb{C}$. Then following statements are easy to prove

(i) If $z_1 \preccurlyeq \max \{z_2, z_3\}$ then $z_1 \preccurlyeq z_2$ if $z_3 \preccurlyeq z_2$;

(ii) If $z_1 \preccurlyeq \max \{z_2, z_3, z_4\}$ then $z_1 \preccurlyeq z_2$ if $\max \{z_3, z_4\} \preccurlyeq z_2$;

(iii) If $z_1 \preccurlyeq \max \{z_2, z_3, z_4, z_5\}$ then $z_1 \preccurlyeq z_2$ if $\max \{z_3, z_4, z_5\} \preccurlyeq z_2$, and so on.

Definition 2.6: Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is a natural number n_0 such that for all $n > n_0$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent sequence. We denote this by $\lim_n x_n = x$ or $x_n \rightarrow x$ as $n \rightarrow \infty$.

Definition 2.7: Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is a natural number n_0 such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called a Cauchy sequence in (X, d).

Definition 2.8: If every Cauchy sequence is convergent in X then (X, d) is called a complete complex valued metric space.

Definition 2.9: Let X be a non-empty set and T: X → X be a self map. Then x ∈ X is a **fixed point** of T if T(x) = x. We denote by Fix (T), the set of all fixed points of T.

Definition 2.10: Let X be a metric space and T, S: X → X. A point x ∈ X is called

(1) a **coincidence point** of the pair (T, S) if Tx = Sx,

(2) a **common fixed point** of the pair (T, S) if x = Tx = Sx.

Definition 2.11: Let (X, d) be a metric space and M be a non empty subset of X. Let T, S: X → X. The pair {S, T} is said to be **weakly compatible**if they commute at their coincidence points, i.e.,if STx = TSx whenever Sx = Tx .

Example 2.12: Let X = $\mathbb{C}$. Define complex metric d: X ×X → $\mathbb{C}$ by $d(z_1, z_2) = e^{ia}|z_1 - z_2|$, where a is any real constant. Then (X, d) is a complex valued metric space. Suppose S, T: X → X be defined as

$$Sz = \begin{cases} 2e^{i\pi/4} & \text{if } \mathrm{Re}(z) \neq 0, \\ 3e^{i\pi/3} & \text{if } \mathrm{Re}(z) = 0. \end{cases} \text{ and } Tz = \begin{cases} 2e^{i\pi/4}, & \text{if } \mathrm{Re}(z) \neq 0, \\ 4e^{i\pi/6}, & \text{if } \mathrm{Re}(z) = 0. \end{cases}$$

Then S and T are coincident when $\mathrm{Re}(z) \neq 0$ and $Sz = Tz = 2e^{i\pi/4}$. At this point $TSz = STz = 2e^{i\pi/4}$. Hence the pair {S, T} commutes at their coincidence point. Therefore, it is weakly compatible at all z ∈$\mathbb{C}$ with $\mathrm{Re}(z) \neq 0$.

Definition 2.13: Let A be a subset of the set ($\mathbb{C}$, ≼). If there exists an element u of $\mathbb{C}$ such that z ≼ u for all z in A, then A is bounded above and u is an upper bound. Similarly, if there exists l ∈$\mathbb{C}$ such that l ≼ z for all z in A, then A is bounded below and l is lower bound.

Definition 2.14: For a subset A ⊆ $\mathbb{C}$which is bounded above if there exists an upper bound s of A such that, for every upper bound u of A, s ≼ u, then the upper bound s is called supA. Similarly, for a subset A ⊆ $\mathbb{C}$which is bounded below if there exists a lower bound t of A such that for every lower bound l of A, l ≼ t, then the lower bound t is called inf A.

Remark 2.15:(i) Suppose that $A \subseteq \mathbb{C}$ is bounded above. Then there exists $q = u + iv \in C$ such that $z = x + iy \precsim q = u + iv$, for all $z \in A$. It follows that $x \precsim u$ and $y \precsim v$, for all$z = x + iy \in A$; that is, $S = \{x: z = x + iy \in A\}$ and $T = \{y: z = x + iy \in A\}$ are two sets of real numbers which are bounded above. Hence both sup S and Sup T exist. Let $x^* =$ sup S and $y^* =$ sup T. Then $z^* = x^* + iy^*$ is supA. Similarly, if $A \subseteq \mathbb{C}$ is bounded below, then $z' = x' + iy'$ is inf A, where $x' = \inf S = \inf\{x : z = x + iy \in A\}$ and $T = \inf\{y : z = x + iy \in A\}$

(ii) Any subset $A \subseteq \mathbb{C}$ which is bounded above has supremum. Equivalently, any subset $A \subseteq \mathbb{C}$ which is bounded below has infimum.

Definition 2.16: Let (X, d) be a complex valued metric space and M be any closed subset of X. If there exists a $z_0 \in M$ such that $d(z, z_0) = d(z, M) = \inf_{z_1 \in M} d(z, z_1)$then z_0 is called a **best approximation to z out of M**. We denote by $\mathbf{P_M(z)}$, the set of all best approximation to z out of M.

3. Main Results:

Theorem 3.1: Let (X, d) be a complex valued metric space and M be a subset of X. Let f, g, S and T be four self maps of X and u be common fixed point of f, g, S, T. If $D = P_M(u)$ and f, g, S and T satisfy (1.2) for all x, y in $D \cup \{u\}$. If $T(\partial M) \subseteq M$, $S(\partial M) \subseteq M$, fD = D, gD = D and D or fD or gD is complete. If the pairs {T, g} and {S, f} are weakly compatible. Then f, g, S and T have a unique common fixed point in $P_M(u)$.

Proof: Let $y \in D$ then $fy \in D$ and $gy \in D$. By the definition of $P_M(u)$, $y \in \partial M$. Since $T(\partial M) \subseteq M$ and $S(\partial M) \subseteq M$, it follows that Sy, $Ty \in M$. Now

$d(Sy, u) = d(Sy, Tu)$

$\precsim a d(fy, gu) + b\,[d(fy,Sy) + d(gu,Tu)] + c\,[d(fy, Tu) + d(gu, Sy)]$ by (2.1)

$= a d(fy, u) + b\,[d(fy,Sy)] + c\,[d(fy, u) + d(u, Sy)]$

$= (a + c)d(fy, u) + b\,[d(fy,Sy)] + c\,[d(u, Sy)]$

$\precsim (a + c)d(fy, u) + b\,[d(fy,u) + d(u, Sy)] + c\,[d(u, Sy)]$

$= (a + b + c)d(fy, u) + (b + c)\,[d(u, Sy)]$.

This implies that

$d(Sy, u) \preccurlyeq \frac{a+b+c}{1-b-c} d(fy, u)$.

Now since $\frac{a+b+c}{1-b-c} < 1$, therefore $d(Sy, u) \preccurlyeq d(fy, u)$.

Hence $Sy \in M$ and $fy \in D$ implies that $Sy \in D$. Similarly $Ty \in D$. Thus f, g, S and T are four self maps of D. Therefore by Theorem 1.2, there exists a unique $z \in D$ such that z is common fixed point of f, g, S and T.

Theorem 3.2: Let (X, d) be a complex valued metric space, M be a subset of X. Let F, G, I, J, K and L be self-maps of X and u be common fixed point of F, G, I, J, K and L. If $D = P_M(u)$, F, G, I, J, K and L satisfy (1.3) for all x, y in $D \cup \{u\}$ and $IJ(\partial M) \subseteq M$, $KL(\partial M) \subseteq M$, $F(D) = D$ and $G(D) = D$. If D or F(D) or G(D) is complete, the pairs {F, IJ} and {G, KL} are weakly compatible and the pairs {K, L}, {K, G}, {L, G}, {I, J}, {I, F}, {J, F} are commuting. Then F, G, I, J, K and L have a unique common fixed point in $P_M(u)$.

Proof: Let $y \in D$ then $Fy \in D$ and $Gy \in D$. By the definition of $P_M(u)$, $y \in \partial M$. Since $IJ(\partial M) \subseteq M$ and $KL(\partial M) \subseteq M$, it follows that IJy, $KLy \in M$. Now,

$d(IJy, u) = d(IJy, KLu)$

$\preccurlyeq ad(Fy, Gu) + b\,[d(Fy, IJy) + d(Gu, KLu)] + c\,[d(Fy, KLu) + d(Gu, IJy)]$ by (1.3),

$= ad(Fy, u) + b\,[d(Fy, IJy)] + c\,[d(Fy, u) + d(u, IJy)]$,

$= (a + c)d(Fy, u) + b\,[d(Fy, IJy)] + c\,[d(u, IJy)]$,

$\preccurlyeq (a + c)d(Fy, u) + b\,[d(Fy,u) + d(u, IJy)] + c\,[d(u, IJy)]$,

$= (a + b + c)d(Fy, u) + (b + c)\,[d(u, IJy)]$

This implies that

$d(IJy, u) \preccurlyeq \frac{a+b+c}{1-b-c} d(Fy, u)$.

Now since $\frac{a+b+c}{1-b-c} < 1$, therefore $d(IJy, u) \preccurlyeq d(Fy, u)$.

Hence IJy ∈ M and Fy ∈ D implies that IJy ∈ D. Similarly KLy ∈ D. Thus F, G, I, J, K and L are four self maps of D. Therefore by Theorem 1.3, there exists a unique z ∈ D such that z is common fixed point of F, G, I, J, K and L.

Theorem 3.3: Let (X, d) be a complete complex valued metric space, M be a subset of X. Let f, g, S and T be four self maps of X and u be common fixed point of f, g, S, T. If $D = P_M(u)$, and f, g, S and T satisfy (1.4) for all x, y in D∪ {u} and $T(\partial M) \subseteq M$, $S(\partial M) \subseteq M$, fD = D and gD = D. If D or fD or gD is complete, $T(P_M(u))$ is closed and the pair {T, g} and {S, f} are weakly compatible, then f, g, S and T have a unique common fixed point in $P_M(u)$.

Proof: Let y ∈ D then fy ∈ D and gy ∈ D. By the definition of $P_M(u)$, $y \in \partial M$. Since $T(\partial M) \subseteq M$ and $S(\partial M) \subseteq M$, it follows that Sy, Ty ∈ M. Now

d(Sy, u) = d(Sy, Tu)

$\preccurlyeq$ max $\{d(fy, gu), \frac{d(fy,Sy)+d(gu,Tu)}{2}, \frac{d(fy,Tu)+d(gu,Sy)}{2}\}$ by (1.4)

$=$ max $\{d(fy, u), \frac{d(fy,Sy)}{2}, \frac{d(fy,u)+d(u,Sy)}{2}\}$

$\preccurlyeq$ max $\{d(fy, u), \frac{d(fy,u)+d(u,Sy)}{2}, \frac{d(fy,u)+d(u,Sy)}{2}\}$

= max $\{d(fy, u), \frac{d(fy,u)+d(u,Sy)}{2}\}$

$\preccurlyeq$d(fy, u).

Thus d(Sy, u) $\preccurlyeq$d(fy, u).

Hence Sy ∈ M and fy ∈ D implies that Sy ∈ D. Similarly, Ty ∈ D. Thus f, g, S and T are four self maps of D. Therefore by Theorem 1.4, there exists a unique z ∈ D such that z is common fixed point of f, g, S and T.

Now, we give an example in support of above result:

Example 3.5: Let X= [0,3] be complex valued metric space with d(x, y) =| x -y |i and M = [1, 2].Let f, g,Sand Tbe self-maps of X definedby:

$$Sx = \begin{cases} 0, & \text{if } x = 0 \\ \frac{2}{3}, & \text{if } x \in (0,1) \\ 1, & \text{if } x \in [1,3] \end{cases}$$

$$Tx = \begin{cases} 0, & \text{if } x = 0 \\ \frac{1}{2}, & \text{if } x \in (0,1) \\ 1, & \text{if } x \in [1,3] \end{cases}$$

$$fx = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x = 1 \\ \frac{3}{2}, & \text{if } x \in (0,1) \\ \frac{x}{3}, & \text{if } x \in (1,3] \end{cases}$$

$$gx = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x = 1 \\ \frac{4}{3}, & \text{if } x \in (0,1) \\ \frac{x}{2}, & \text{if } x \in (1,3] \end{cases}$$

Clearly, $S(X) = \{0, 1, \frac{2}{3}\} \subseteq \{0\} \cup (\frac{1}{2}, \frac{3}{2}] = g(X)$ and $T(X) = \{0, 1, \frac{1}{2}\} \subseteq \{0, \frac{3}{2}\} \cup (\frac{1}{3}, 1] = f(X)$. Also, $u = 0$, $D = P_M(0) = \{1\}$, $T(\partial M) \subseteq M$, $S(\partial M) \subseteq M$, $fD = D$ and $gD = D$. Further, D is complete, $T(P_M(u))$ is closed, the pairs (f, k) and (g,h) are weaklycompatibleand f, g, S and T satisfy (1.4) for all x, y in $D \cup \{u\}$. Clearly, '1' is the unique common fixed point of f, g,hand k in $P_M(u)$.

References:

1. Azam, A., Fisher, B. and Khan M.: Common fixed point theorems in complex valued metric spaces, Numerical Functional Analysis and Optimization, 32 (3), (2011), 243-253.
2. Bhatt, S., Chaukiyal, S. and Dimri, R. C.: A common fixed point theorem for weakly compatible maps in complex valued metric spaces, International Journal of Mathematical Sciences and Applications, 1(3), (2011), 1385-1389.
3. Chistyakov, V.V.: Modular metric spaces, I: basic concepts. Nonlinear Anal., 72, (2010), 1–14.
4. Ghaler, S.: 2-metrische raume und ihre topologische strukture, math. Nachr, 26, (1963), 115-148.
5. Huang, L.G. and Zhang, X.: Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl., 332, (2007), 1468–1476.
6. Mustafa, Z. and Sims, B.: A new approach to generalized metric spaces. J Nonlinear Convex Anal., 7(2), (2006), 289–297.
7. Perov, AI: On the Cauchy problem for a system of ordinary difierential equations. Pvi-blizhen Met Reshen Diff Uvavn., 2, (1964),115–134.
8. Sastry, K. P. R, Naidu, G. A, Bekeshie, Tadesse and Rahamatulla, M. A.: A common fixed point theorem for four self maps in complex valued and vector valued metric spaces, International Journal of Mathematical Archive, 3(7), (2012), 2680 – 2685.
9. Tiwari, R. and Shukla, D.P., Six maps with a common fixed point in complex valued metric spaces, International Journal of Mathematical Sciences and Applications,2(2), (2012), 827 – 832.