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On P_4-Decomposition of Some Graphs

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Abstract:

Let G = (V, E) be a connected simple graph of order p and size q. If $G_1, G_2, ..., G_n$ are edge disjoint subgraphs of G such that $E(G) = E(G_1) \cup E(G_2) \cup ... \cup E(G_n)$, then $(G_1, G_2, ..., G_n)$ is said to be Decomposition of G. In this paper, we study the P_4-decomposition of some graphs.

Keywords: Decomposition.

Subject classification: 05C99

Introduction:

Let G be a finite, simple, undirected (p, q) graph with vertex set V(G) and edge set E(G). For graph theoretic terminology refer to Harary [2], Buckley and Harary [1].Let G = (V, E) be a connected simple graph of order p and size q. If $G_1, G_2, ..., G_n$ are edge disjoint subgraphs of G such that $E(G) = E(G_1) \cup E(G_2) \cup ... \cup E(G_n)$, then $(G_1, G_2, ..., G_n)$ is said to be Decomposition of G.A graph $H = (V_1, X_1)$ is called a subgraph of G = (V, X) if $V_1 \subseteq V$ and $X_1 \subseteq X$. H is called an induced subgraph of G if H is the maximal subgraph of G with point set V_1.

Definition: A graph is a complete graph if every pair of its vertices are adjacent. A complete graph with *n* vertices is denoted by K_n.

Definition: The complement $\overline{G}$ of a graph G is that graph whose vertex set is V(G) and for each pair *u*, *v* of vertices of G, *uv* is an edge of $\overline{G}$ if and only if *uv* is not an edge of G.

Definition:A Complete bipartite graph is a bipartite graph with vertex partition V_1 and V_2 such that each that each vertex of V_1 is joined to every vertex of V_2. If V_1 contains *m* vertices and V_2 contains *n* vertices, then the complete bipartite graph is denoted by $K_{m,n}$.

Definition: The corona $G_1 \circ G_2$ of two graphs G_1 and G_2 is the graph obtained by taking one copy of G_1 of order*n* and *n* copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.

In 2003,Sunil Kumar studied the P_4-decomposition of graphs [4].

Theorem:1.1[4] K_n is P_4-decomposable if and only if n > 3 and $n \equiv 0, 1 \pmod 3$.

Theorem:1.2[4] $K_{m,n}$ is P_4-decomposable if and only if $m \geq 2$, $n \geq 2$ and $mn \equiv 0 \pmod 3$.

Theorem:1.3[4] If n $\equiv$ 2(mod 3) and n > 4, then $K_n - e$ is P_4-decomposable.

Theorem:1.4[4] If n $\equiv$ 0(mod 3), then $K_{2n} - F$ is P_4-decomposable, where F is a 1-factor of K_{2n}.

Theorem:1.5[4]The graph obtained by attaching a pendant vertex to alternative vertices in C_{2n} is P_4-decomposable.

Theorem:1.6[4]The graph obtained by joining odd(even) alternative vertices in C_{2n} is P_4-decomposable.

Theorem:1.7[3] (Bermond) (i) If *p* is even, K_p can be decomposed into p/2 Hamiltonian paths (ii) If *p* is odd, K_p can be decomposed into (p–1)/2 Hamiltonian cycles and C_{2n}.

If *p* is even, K_p can be decomposed into ((p–2)/2) cycles of length nand (p/2)K_2's.

2. P_4-Decomposition of Some Graphs

In this section, we prove the following theorems:

(i)$K_n \circ 2K_1$ is P_4-decomposable if and only if n $\equiv$0(mod 3).

(ii) $K_n \circ K_3$ is P_4-decomposable into $\frac{n(n+11)}{6}$ times P_4if and only if n $\equiv$0, 1(mod 3).

(iii)$K_n \circ K_2$ is P_4-decomposable if and only if n $\equiv$0, 1(mod 3).

To prove this, we need the following lemma's.

Lemma 2.1 Whenn $\equiv$ 0 (mod 3), $\overline{C_n}$ is decomposable into P_4's.

Proof: Case (i)*n* is odd

When*n* is odd, K_n is decomposable into $\frac{n(n-1)}{2}$ C_n's. Hence, $\overline{C_n}$ is decomposable into $\frac{n(n-3)}{2}$ C_n's. When n is a multiple of three each C_n is decomposable into $\frac{n}{3}$ P_4's. Hence, $\overline{C_n}$ is decomposable into $\frac{n(n-3)}{6}$ P_4's.

Case (ii)*n* is even

When*n* is even, K_n is decomposable into $\frac{(n-2)}{2}$ C_n'sand $\frac{n}{2}$ K_2's. Therefore, $\overline{C_n}$ is decomposable into $\frac{(n-4)}{2}$ C_n'sand $\frac{n}{2}$ K_2's. Now $\frac{(n-6)}{2}$ C_n's are decomposed into $\frac{n(n-6)}{6}$

P_4'sand a C_n with $\frac{n}{2}$ K_2'scan be decomposed into $\frac{n}{2}$ P_4's. Hence, $\overline{C_n}$ is decomposable into $\frac{n(n-3)}{6}$ P_4's.

Lemma 2.2 Ifn $\equiv$ 2 (mod 3), then $\overline{C_n}$ – {e_1, e_2} is P_4 decomposable.

Proof: Let G= $\overline{C_n}$, where $\overline{C_n}$ = K_n–C_n.

$$E(G - \{e_1, e_2\}) = E(\bar{C}_n) - \{e_1, e_2\}$$

$$= E(K_n - e_1) - E(C_n + e_2)$$

Since n $\equiv$ 2 (mod 3), then by Theorem 1.3,K_n–e is P_4-decomposable andC_n + e is also P_4 decomposable. Therefore G – {e_1, e_2} is P_4 decomposable. Hence $\overline{C_n}$ – {e_1, e_2} is P_4 decomposable.

Lemma 2.3 Ifn $\equiv$ 2 (mod 3), then$K_{n,n}$– e is P_4-decomposable.

Proof: LetV_1,V_2 be the vertices of V. Letu_1, u_2, …,$u_n \in V_1$ and v_1, v_2, …, $v_n \in V_2$. Let e = $u_n v_n$. Then edges of $K_{n,n}$– e can be decomposed into edges of $K_{n-2, n-2}$, edges of $K_{2, n-2}$, edges of $K_{2, n-2}$ and $K_{2, 2}$– e.

$$K_{n-2, n-2} = \langle u_1, u_2, \ldots, u_{n-2}, v_1, v_2, \ldots, v_{n-2} \rangle,$$

$$K_{2, n-2} = \langle u_{n-1}, u_n, v_1, v_2, \ldots, v_{n-2} \rangle,$$

$$K_{2, n-2} = \langle u_1, u_2, \ldots, u_{n-2}, v_{n-1}, v_n \rangle \text{ and } K_{2, 2} - e = \langle u_{n-1}, u_n, v_{n-1}, v_n \rangle.$$

We know by Theorem 1.3, since n–2 $\equiv$ 0(mod 3), $K_{n-2, n-2}$ is P_4-decomposable.$K_{2, n-2}$is P_4-decomposable by Theorem 1.3,since 2(n– 2) $\equiv$0(mod 3) and $K_{2, 2}$–e = P_4. Hence$K_{n, n}$– e is P_4 decomposable, when n $\equiv$ 2 (mod 3).

Lemma 2.4 Ifn $\equiv$ 0 (mod 3), then$K_{n,n}$–F is P_4 decomposable, where F is a 1-factor of $K_{n,n}$.

Proof:$K_{n,n}$– F has n(n–1) edges. Let G = $K_{3, 3}$– 1 factor. $K_{n, n}$– F can be decomposed into n/3 G's and ((n/3)–1)$K_{3,3}$. G can be decomposed into2 P_4's. n/3 G's can be decomposed into2n/3 P_4's.$K_{3,3}$can be decomposed into3 P_4's. ((n/3)–1)$K_{3,3}$can be decomposed into 3((n/3)–1)P_4's.Therefore, $K_{n, n}$– F can be decomposed into (n(n–1))/3 P_4's. Hence, $K_{n,n}$– F is P_4 decomposable.

Lemma 2.5 Let $G = K_{n,n} - F$, where F is a one factor of $K_{n,n}$. If $n \equiv 2 \pmod 3$, then $G - \{e_1, e_2\}$ is P_4 decomposable, where e_1, e_2 are non-adjacent edges of G.

Proof: Let V_1, V_2 be the vertices of V. Let $u_1, u_2, \ldots, u_n \in V_1$ and $v_1, v_2, \ldots, v_n \in V_2$. Let $G = K_{n,n} - F$. G can be decomposed into edges of $K_{2, n-2}$, edges of $K_{2, n-2}$, edges of $K_{n-2, n-2} - F_1$, where F_1 is a 1-factor of $K_{n-2, n-2}$, and $2K_2$ or $\{e_1, e_2\}$.

$K_{n-2, n-2} - F_1$ has $((n-2)(n-3))$ edges.

$K_{2, n-2} = \langle u_1, u_2, v_3, v_4, \ldots, v_n\rangle$ is decomposable into $(2n - 4)/3 P_4$'s.

$K_{2, n-2} = \langle u_3, u_4, \ldots, u_n, v_1, v_2\rangle$ is decomposable into $(2n - 4)/3 P_4$'s.

$K_{n-2, n-2} - F_1 = \langle u_3, u_4, \ldots, u_n, v_3, v_4, \ldots, v_n\rangle - \{u_i v_i / i = 3, 4, \ldots n\}$ can be decomposed into $((n-2)(n-3))/3 P_4$'s.

$2K_2 = \langle u_1, u_2, v_1, v_2\rangle - \{u_1v_1, u_2v_2\}$, $e_1 = u_1v_2$, $e_2 = u_2v_1$

Therefore, $G - \{e_1, e_2\}$ can be decomposed into $2K_{2, n-2}$, $K_{n-2, n-2} - F$. Hence, $G - \{e_1, e_2\}$ is P_4 decomposable and e_1, e_2 are not adjacent when $n \equiv 2 \pmod 3$.

Lemma 2.6 Let $G = (K_{n,n} + e) - F$, where F is a one factor of $K_{n,n}$. If $n \equiv 2 \pmod 3$, then G is P_4 decomposable.

Proof: Let $u_1, u_2, \ldots, u_n \in V_1$ and $v_1, v_2, \ldots, v_n \in V_2$. Let $e = u_1u_2$ or v_1v_2. $G = (K_{n,n} + e) - F$ can be decomposed into $K_{2, n-2}$, $K_{2, n-2}$, $K_{n-2, n-2}$ and $2K_2 \cup e = P_4$.

$K_{2, n-2} = \langle u_1, u_2, v_3, v_4, \ldots, v_n\rangle$ is decomposable into $(2n - 4)/3 P_4$'s.

$K_{2, n-2} = \langle u_3, u_4, \ldots, u_n, v_1, v_2\rangle$ is decomposable into $(2n - 4)/3 P_4$'s.

$K_{n-2, n-2} = \langle u_3, u_4, \ldots, u_n, v_3, v_4, \ldots, v_n\rangle - \{u_i v_i / i = 3, 4, \ldots n\}$ can be decomposed into $((n-2)(n-3))/3\ P_4$'s.

$2K_2 \cup e = \langle u_1, u_2, v_1, v_2, e\rangle - \{u_1v_1, u_2v_2\}$

Since, $n \equiv 2 \pmod 3$, $K_{2,n-2}$ is P_4-decomposable by Theorem 1.3 and $K_{n-2, n-2}$ is also P_4-decomposable by Theorem 1.2., $2K_2 \cup e$ form a P_4. Hence, $(K_{n,n} + e) - F$ is P_4 decomposable when $n \equiv 2 \pmod 3$.

Lemma 2.7 $K_{n,n} - C_{2n}$ is P_4 decomposable if and only if $n \equiv 0, 2 \pmod 3$.

Proof: Case(i) $n \equiv 0 \pmod 3$.

If $n \equiv 0 \pmod 3$. $K_{n,n}$ is P_4-decomposable by Theorem1.2 and C_{2n} is also P_4-decomposable. Hence, $K_{n,n} - C_{2n}$ is P_4-decomposable.

Case(ii) $n \equiv 2 \pmod 3$.

If $n \equiv 2 \pmod 3$. $K_{n,n}$ is not P_4-decomposable. $E(K_{n,n} - C_{2n}) = E(K_{n,n} - e) - E(P_{2n})$. P_{2n} has $2n-1$ edges which is a multiple of 3. Therefore, P_{2n} is P_4-decomposable. Since $n \equiv 2 \pmod 3$, by Lemma 2.3., $K_{n,n} - e$ is P_4-decomposable. Hence, $K_{n,n} - C_{2n}$ is P_4-decomposable.

Conversely, assume that $K_{n,n} - C_{2n}$ is P_4-decomposable. Then $|K_{n,n} - C_{2n}| \equiv 0 \pmod 3$. Number of edges in $K_{n,n} - C_{2n} = n^2 - 2n$.

This implies that, $n^2 - 2n \equiv 0 \pmod 3$, $n(n-2) \equiv 0 \pmod 3$. Hence, $n \equiv 0, 2 \pmod 3$.

Theorem 2.1 $K_n \circ 2K_1$ is P_4-decomposable if and only if $n \equiv 0 \pmod 3$.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n. Let v_i', v_i'' be the vertices i^{th} copy of $2K_1$. Therefore, $V(K_n \circ 2K_1) = \{v_1, v_2, \ldots, v_n, v_1', v_2', \ldots, v_n', v_1'', v_2'', \ldots, v_n''\}$ be the vertices of $K_n \circ 2K_1$. $K_n \circ 2K_1$ has $3n$ vertices and $n(n+3)/2$ edges. $K_n \circ 2K_1$ can be decomposed into $C_n \circ 2K_1$ and $\overline{C_n}$ When $n \equiv 0 \pmod 3$, $\overline{C_n}$ is P_4-decomposable by Lemma2.1. $C_n \circ 2K_1$ can be decomposed into $\langle v_i'', v_i, v_{i+1}, v_{i+1}' \rangle$, $i = 1, 2, \ldots, n$ and $\langle v_n'', v_n, v_1, v_1' \rangle$. Hence, $K_n \circ 2K_1$ is P_4-decomposable.

Conversely, assume that $K_n \circ 2K_1$ is P_4-decomposable. Then $|K_n \circ 2K_1| \equiv 0 \pmod 3$. Number of edges in $K_n \circ 2K_1 = \frac{n(n+3)}{2}$. This implies that, $\frac{n(n+3)}{2} \equiv 0 \pmod 3$. Hence, $n \equiv 0 \pmod 3$.

Theorem 2.2 $K_n \circ K_3$ is P_4-decomposable into $\frac{n(n+11)}{6}$ times P_4 if and only if $n \equiv 0, 1 \pmod 3$.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n. Let v_i', v_i'', v_i''' be the vertices i^{th} copy of K_3. Therefore, $V(K_n \circ K_3) = \{v_1, v_2, \ldots, v_n, v_1', v_2', \ldots, v_n', v_1'', v_2'', \ldots, v_n'', v_1''', v_2''', \ldots, v_n'''\}$ be the vertices of $K_n \circ K_3$. $K_n \circ K_3$ has $4n$ vertices, $(n(n+11))/2$ edges. K_n is decomposable into P_4 if and only if $n \equiv 0, 1 \pmod 3$ by Theorem1.1 and nK_4 is always P_4-decomposable.

Theorem 2.3 If $n \equiv 1 \pmod 3$ and n is even, then $\overline{C_n} - F$ can be decomposed into

$\frac{n(n-4)}{6}$ times P_4's.

Proof: Since n is even. $\overline{C_n}$ can be decomposed into 1-factor and $(n(n-4))/2C_n$'s. Therefore, $\overline{C_n} - F$ can be decomposed into $(n(n-4))/2C_n$'s. Since n is even and $n \equiv 1 \pmod 3$, $(n-4)/2$ is a multiple of 3. These (n–4)/2 cycles can be combined into (n–4)/6 times $3C_n$'s. Since C_n's are spanning cycles we can decompose $3C_n$'s into nP_4's by Lemma 2.8. Hence, (n–4)/6 ($3C_n$'s) can be decomposed into (n(n–4))/6 P_n's.

Theorem 2.4 $K_{nn} - C_{2n} - \{e_1, e_2\}$ is P_4-decomposable if and only if $n \equiv 1 \pmod 3$.

Proof: Let $V_1 = \{x_1, x_2, \ldots, x_n\}$, $V_2 = \{y_1, y_2, \ldots, y_n\}$ be the partitions of $V(K_{n,n})$.

Let $x_1, y_1, x_2, y_2, x_3, y_3, \ldots, x_{n-2}, y_{n-2}, x_{n-1}, y_{n-1}, x_n, y_n, x_1$ be a C_{2n} in $K_{n,n}$. Now, K_{nn} can be decomposed into $K_{n-3, n-3}$, $K_{3, n-3}$, $K_{3, n-3}$ and $K_{3, 3}$ as follows.

$$K_{n-3, n-3} = \langle x_1, x_2, \ldots, x_{n-3}, y_1, y_2, \ldots, y_{n-3}\rangle$$

$$K_{3, n-3} = \langle x_1, x_2, \ldots, x_{n-3}, y_{n-2}, y_{n-1}, y_n\rangle$$

$$K_{3, n-3} = \langle y_1, y_2, \ldots, y_{n-3}, x_{n-2}, x_{n-1}, x_n\rangle$$

$$K_{3, 3} = \langle x_{n-2}, x_{n-1}, x_n, y_{n-2}, y_{n-1}, y_n\rangle$$

So $K_{nn} - C_{2n}$ can be decomposed into

$$\langle x_1, x_2, \ldots, x_{n-3}, y_1, y_2, \ldots, y_{n-3}\rangle - \langle x_1, y_1, x_2, y_2, \ldots, x_{n-3}, y_{n-3}\rangle = K_{n-3, n-3} - P_{2(n-3)}.$$

$$\langle x_1, x_2, \ldots, x_{n-3}, y_{n-2}, y_{n-1}, y_n\rangle = K_{n-3, 3} - \{y_n, x_1\} = (K_{n-3, 3} - e_1)$$

$$\langle y_1, y_2, \ldots, y_{n-3}, x_{n-2}, x_{n-1}, x_n\rangle - \{y_{n-3}, x_{n-2}\} = K_{3, n-3} - e$$

$$\langle x_{n-2}, x_{n-1}, x_n, y_{n-2}, y_{n-1}, y_n\rangle - \{x_{n-2}, y_{n-2}, x_{n-1}, y_{n-1}, x_n, y_n\} = K_{3, 3} - P_6 = P_4 \cup K_2$$

Lemma 2.8 Let C_1, C_2, C_3 be any tree spanning cycles on n vertices $v_1, v_2, \ldots, v_n$, then edges of $C_1 \cup C_2 \cup C_3$ can be decomposed into n times P_4's.

Example: C_9 can be decomposed into 3 cycles as follows:

C_1: $v_0, v_1, v_2, v_8, v_3, v_7, v_4, v_6, v_5, v_0$;

C_2: $v_0, v_2, v_3, v_1, v_4, v_8, v_5, v_7, v_6, v_0$;

C_3: $v_0, v_3, v_4, v_2, v_5, v_1, v_6, v_8, v_7, v_0$.

The edges $\{v_0v_1\}$ from C_1, $\{v_1v_4\}$ from C_2 and $\{v_4v_2\}$ from C_3 form a path P_4. Continuing this process, we get $\{v_0\, v_1 v_4\, v_2,\ v_1\, v_2\, v_3\, v_4,\ v_2\, v_8\, v_5\, v_1,\ v_8\, v_3\, v_1\, v_6,\ v_3\, v_7\, v_6\, v_8,\ v_7\, v_4\, v_8\, v_7,\ v_4\, v_6\, v_0\, v_3,\ v_6\, v_5\, v_7\, v_0,\ v_5\, v_0\, v_2\, v_5\}$. Hence, $C_1 \cup C_2 \cup C_3$ can be decomposed into nP_4's.

Theorem 2.5 $K_n \circ K_2$ is P_4-decomposable if and only if $n \equiv 0, 1 \pmod 3$.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n. Let v_i', v_i'' be the vertices i^{th} copy of K_2. Let $V(K_n \circ K_2) = \{v_1, v_2, \ldots, v_n, v_1', v_2', \ldots, v_n', v_1'', v_2'', \ldots, v_n''\}$ be the vertices of $K_n \circ K_2$. $K_n \circ K_2$ has 3n vertices, $(n(n+5))/2$ edges.

Case(i) n is even, $n \equiv 0, 1 \pmod 3$.

Consider a C_n from K_n and a 1-factor F from K_n. Combining this with nK_3 we can decompose C_n, F and nK_3 into $3n/2$ P_4's.

Remaining edges contains $(n-4)/2$ C_n's or $(n(n-4))/2$ edges.

Subcase(i) If $n \equiv 0 \pmod 3$, C_n is decomposable into $n/3$ P_4's and $(n-4)/2$ C_n's is decomposable into $(n(n-4))/6$ P_4's. Hence the theorem.

Subcase(ii) If $n \equiv 1 \pmod 3$, but n is even implies that $n-4$ even. Hence, $(n-4)/2$ is a multiple of 3. By Theorem 2.3., $(n-4)/2$ cycles can be decomposed into $(n(n-4))/6$ times P_4. Hence, $K_n \circ K_2$ can be decomposed into $(3n/2) + ((n(n-4))/6) = (n(n+5))/6$ times P_4.

Case(ii) n is odd, $n \equiv 0, 1 \pmod 3$

K_n can be decomposed into $(n-1)/2C_n$'s. Consider $3C_n$'s and nK_3's. They can be decomposed into 2n times P_4's. ---------- (1)

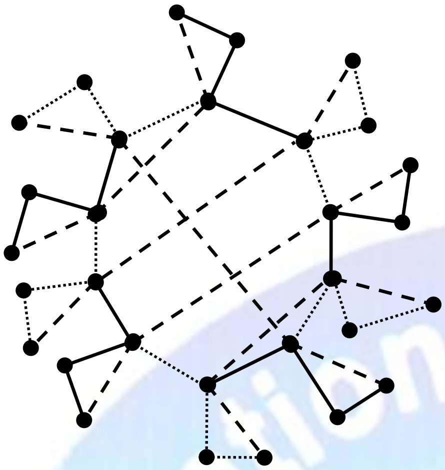
Remaining there are $(n-7)/2C_n$'s.

Subcase (i) If $n \equiv 0 \pmod 3$, each C_n can be decomposed into $n/3$ P_4's. Hence, $(n-7)/2$ C_n's can be decomposed into $n(n-7)/6$ P_4's ---------- (2)

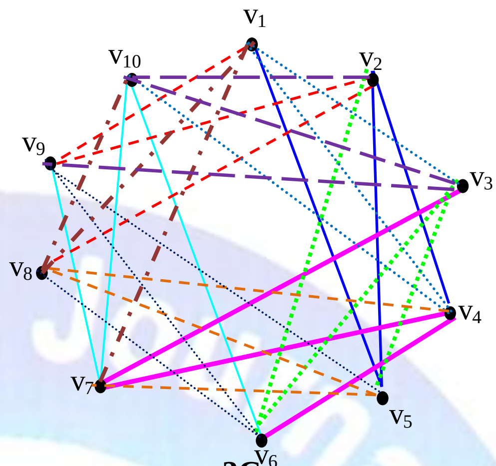
Hence from (1) and (2), $K_n \circ K_2$ can be decomposed into $(n(n+5))/6$ times P_4's.

Subcase (ii) $n \equiv 1 \pmod 3$, $n-7 = 3k$. But n is odd implies that $n-7$ even. Hence, $(n-7)/2$ is a multiple of 3. Hence by the Lemma 2.8., 3 cycles can be decomposed into nP_4's. Hence, $(n-7)/2$ C_n's can be decomposed into $(n^2-7n)/6$ P_4's. Hence, $K_n \circ K_2$ can be decomposed into $(n(n+5))/6$ times P_4.

Example: The graph $K_{10} \circ K_2$ can be decomposed into $\mathbf{C_{10} \cup F \cup 10K_3 \cup 3C_{10}}$



$C_{10} \cup F \cup 10K_3$



$3C_{10}$

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