Vol.09 Issue-01, (January - June, 2017) ISSN: 2394-9309 (E) / 0975-7139 (P)
Aryabhatta Journal of Mathematics and Informatics (Impact Factor- 5.856)

A Sine Trigonometric Similarity Measures on Fuzzy Rough Sets & It's Applications in Medical Diagnosis

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Abstract
The similarity measure has turn into an imperative gizmo to determine similarity between two groups or between two elements. Several similarity measures were insinuated by researchers operating on various tools. In which some are probabilistic in nature and other are non-probabilistic in nature. Here we discuss a non-probabilistic similarity measures proposed by fuzzy rough set approach. We also prove the validity of this proposed measure and discuss it application for medical diagnosis. We also compare proposed measures with existing ones. The proposed measures can provide a useful tactic to measure the similarity between fuzzy rough sets.
Key Words: *fuzzy sets, rough sets, fuzzy rough sets, similarity measures.*

Introduction

In real life situation, the problems associated with medical science, social science, engineering, economics etc. be inclusive of uncertainty, imprecision and vagueness. To deal with these types of challenges conventional mathematical or statistical implements are not enough. Due to the interesting nature of these types of dilemmas, many authors paying attention to modelling uncertainty and have projected various theories recently. Theory of fuzzy sets [1], theory of intuitionistic fuzzy set [2], theory of vague set [5], theory of rough set [3] are some of well-known theories. In trading with the obstacles arising due to the vagueness present in the real world, all these theories are profitable to some level. For the study and illustration of different types of data information such as numerical information, interval-valued information, linguistic information and etc, the concepts such as cardinality, entropy, distance measure and similarity measure which are associated these theories is extensively used.

One of the revolutionary theory known as fuzzy set theory is insinuated by Zadeh [1] in 1965. Due to its potential for operating on uncertainty the theory of fuzzy set has accomplished to great success. Pawalk [3] in 1982 initiated and developed another world-shattering notion known as rough set theory. Although these two theories are operated to manage vague and imprecise information yet their foundations and importance are different. Thus we say that these are diverse and complementary generalizations of set theory. A fuzzy set allocates a membership value other than 0 and 1. A rough set uses three membership functions, a reference set and its lower and upper approximation space. There are huge analyses on the relationship between rough sets and fuzzy sets [6, 4, and 7]. For the combination of rough and fuzzy sets several suggestions have been made. The effect of these analyses directed towards the initiation of the notions of rough fuzzy sets and fuzzy rough sets [8, 9, 10, 11, and 12]. In [13] Yao state that a fuzzy rough set is originating from the approximation of a crisp set in a fuzzy approximation space. It is a duo of fuzzy sets in which the membership of an element is influenced by the degrees of similarity in all those elements in the set.The measurement of uncertainty is a significant subject for the theories pact with uncertainty. For their wide functioning as different area the similarity measure, distance measures, entropy in fuzzy set theory and the relationship among these measures have been significantly studied. To reveal the degree of similarity between two objects, attributes and sets a similarity measure is a significant tool. Due to their wide applications of real world, measures of similarity between fuzzy sets have achieved interest in researchers. Different type of similarity measures anticipated by researcher with their applications of several fields like pattern recognition, decision making, machine learning, data mining, market prediction and image processing. In recent years various measures of similarity between some fuzzy sets have been insinuated and researched [14, 15]. To measure the degree of similarity between fuzzy rough sets Zhang et al [16], recommend similarity measures. Later Niu Qi et al. [17] also intended a new similarity measures on fuzzy rough sets. In this paper, we proposed a sine trigonometric similarity measures between fuzzy rough sets and its elements. Finally, we illustrate the problem of the contexts of medical diagnosis by similarity measures between fuzzy rough sets and also compare with existing ones.The rest of paper is organized as follows. In section 2, we discuss theoretical background related to this paper in which we reviewed related term and concepts. In section 3, some existing measures of similarity are reviewed, some rules which are considered when we give a similarity measure between some fuzzy rough sets and its elements are proposed. In section 4, we propose some new similarity measures between fuzzy rough sets and show their validity. In section 5, the proposed similarity measures are applied to deal with a problem related to medical diagnosis and compare it with existing ones. At last we conclude the paper.

2. Theoretical Ground

In this section we discuss some related concepts and terms of this paper.

Fuzzy Sets: A fuzzy set F on U is characterized by a membership function $\mu_F(x) : U \to [0, 1]$, as $F = \{x, \mu_F(x) : x \in U\}$, where U is Universe of discourse. Then $F^C = \{x, 1 - \mu_F(x) : x \in U\}$. where F^C is the complement of fuzzy set F.

The component-wise representation of fuzzy-set equality and inclusion are as:

$$F = G \iff \mu_F(x) = \mu_G(x), \quad \forall\, x \in U,$$
$$F \subseteq G \iff \mu_F(x) \le \mu_G(x), \quad \forall\, x \in U,$$

In various descriptions of complement, intersection and union of fuzzy sets, we prefer the standard max-min system insinuated by Zadeh [1], in which fuzzy-set operations are outlined component-wise as:

$$\mu_{F^c} = 1 - \mu_F(x),$$
$$\mu_{F\cap G}(x) = min\{\mu_F(x), \mu_G(x)\},$$
$$\mu_{F\cup G}(x) = max\{\mu_F(x), \mu_G(x)\}.$$

A significant feature of fuzzy-set operations is that they are truth-functional. By using the membership functions of the fuzzy sets one can achieve membership functions of the complement, intersection and union of fuzzy sets.

Rough Sets: A brief recall of rough set is given in next definition:

Definition: Let U be a non-empty universe of discourse and R an equivalent relation on U, which is called an indistinguishable relation, $U/R = \{X_1, X_2, \dots\dots, X_n\}$ is all the equivalent class derived from R. $W = (U, R)$ are called an approximation space. $\forall X \subseteq U$, Suppose $\underline{X} = \{x \in U | [x] \subseteq X\}$ and $\overline{X} = \{x \in U | [x] \cap X \ne \emptyset\}$, a set pairs $(\underline{X}, \overline{X})$ are called a rough set in W, and symbolized as $X = (\underline{X}, \overline{X})$; $\underline{X}$ and $\overline{X}$ are the lower approximation and the upper approximation of X on W respecectively.

The strong and weak membership function of a rough set can be characterized by characteristic function of $\underline{X}$ and $\overline{X}$ respectively. Let the membership function of X and R indicated by μ_X and μ_R respectively. Then lower and upper approximations expressed by the following two expressions:

$$\mu_{\underline{R}(X)}(x) = inf\{\mu_X(y) \mid y \in U, (x, y) \in R\},$$
$$\mu_{\overline{R}(X)}(x) = sup\{\mu_X(y) \mid y \in U, (x, y) \in R\}, \qquad (2.1)$$

and

$$\mu_{\underline{R}(X)}(x) = inf\{1 - \mu_R(x, y) \mid y \notin A\},$$
$$\mu_{\overline{R}(X)}(x) = sup\{\mu_R(x, y) \mid y \in A\}, \qquad (2.2)$$

Here definition (2.2) is not described for sets $\emptyset$ and U. In these circumstances, we basically classify $\mu_{\underline{R}(U)}(x) = 1$ & $\mu_{\overline{R}(\emptyset)}(x) = 0, \forall x \in U$. In successive conversation, we will not clearly state these definitions of boundary cases. Based on the two equivalent definitions, lower and upper approximations may be interpreted as follows. An element $x \in \underline{X}$ if any element not in X is not equivalent to x, namely, $\mu_R(x,y) = 0$. An element $x \in \overline{X}$ if any element in X is equivalent to x, namely, $\mu_R(x,y) = 1$. These two aspects is significant in the combination of rough and fuzzy set. For convenience, the strong and weak membership function of a rough set can be precise as:

$$\mu_{\underline{R}(X)}(x) = inf\{\max[\mu_X(y), 1 - \mu_R(x,y)] \mid y \in U\},$$
$$\mu_{\overline{R}(X)}(x) = sup\{\min[\mu_X(y), \mu_R(x,y)] \mid y \in U, (x,y) \in R\}, \qquad (2.3)$$

For reference sets $X \cap Y$ and $X \cup Y$, $\left(\underline{R}(X \cap Y), \overline{R}(X \cap Y)\right)$ and $\left(\underline{R}(X \cup Y), \overline{R}(X \cup Y)\right)$ are the intersection and union of two rough sets $\left(\underline{R}(X), \overline{R}(X)\right)$ and $\left(\underline{R}(Y), \overline{R}(Y)\right)$, respectively. With a reference set X^c, the complement of rough-set is defined by $\left(\underline{R}(X^c), \overline{R}(X^c)\right)$. In disparity between fuzzy sets, rough-set intersection and union are not truth-functional as indicated by the properties:

1. $\underline{R}(X^c) = \left(\overline{R}(X)\right)^c$,
   $\overline{R}(X^c) = \left(\underline{R}(X)\right)^c$,
2. $\underline{R}(U) = U$,
   $\overline{R}(\emptyset) = \emptyset$,
3. $\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y)$,
   $\overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$,
   $\underline{R}(X \cup Y) \supseteq \underline{R}(X) \cup \underline{R}(Y)$,
   $\overline{R}(X \cap Y) \subseteq \overline{R}(X) \cup \overline{R}(Y)$,
4. $\underline{R}(X) \subseteq X$,
   $X \subseteq \overline{R}(X)$,
5. $X \subseteq \underline{R}\left(\overline{R}(X)\right)$,
   $\overline{R}\left(\underline{R}(X)\right) \subseteq X$,
6. $\underline{R}(X) \subseteq \underline{R}\left(\underline{R}(X)\right)$,
   $\overline{R}\left(\overline{R}(X)\right) \subseteq \overline{R}(X)$.

Fuzzy Rough Sets: The approximation of a crisp set in a fuzzy approximation space is called a fuzzy rough set. We exclaim the pair of fuzzy set $\left(\underline{R}(X), \overline{R}(X)\right)$ a fuzzy rough set with reference set $X \subseteq U$. A fuzzy rough set is characterized by a crisp set and two fuzzy sets:

$$\mu_{\underline{R}(X)}(x) = inf\{1 - \mu_R(x,y) \mid y \notin A\},$$
$$\mu_{\overline{R}(X)}(x) = sup\{\mu_R(x,y) \mid y \in A\},$$

Based on the properties of rough sets, one can see that fuzzy rough sets satisfy the properties: for $X, Y \subseteq U$,

1. $\underline{R}(X^c) = \left(\overline{R}(X)\right)^c$,
   $\overline{R}(X^c) = \left(\underline{R}(X)\right)^c$,

2. $\underline{R}(U) = U$,
   $\overline{R}(\emptyset) = \emptyset$,

3. $$\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y),$$
$$\overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y),$$
$$\underline{R}(X \cup Y) \supseteq \underline{R}(X) \cup \underline{R}(Y),$$
$$\overline{R}(X \cap Y) \subseteq \overline{R}(X) \cup \overline{R}(Y),$$
4. $$\underline{R}(X) \subseteq X,$$
$$X \subseteq \overline{R}(X),$$

For fuzzy rough sets, we do not have properties similar to 5 and 6 described in rough sets. From above conversation we classify fuzzy rough sets as:

Definition: Let S be the set of the whole rough sets, $A = (\underline{A}, \overline{A}) \in S$, then a fuzzy rough set $X = (\underline{X}, \overline{X})$ in A can be expressed by a pair mapping $\mu_{\underline{X}},\ \mu_{\overline{X}}$

$$\mu_{\underline{X}}\ :\ \underline{X} \to [0,1], \qquad \mu_{\overline{X}}\ :\ \overline{X} \to [0,1].$$

Also $\mu_{\underline{X}} \leq\ \mu_{\overline{X}},\ \forall x \in \dot{\overline{X}}$. And then, a fuzzy rough set X in A could be signified by

$$X = \{\langle x, \mu_{\underline{X}},\ \mu_{\overline{X}}\rangle\ |\forall x \in \overline{X}\}. \qquad (2.4)$$

and $\{\langle \mu_{\underline{X}},\ \mu_{\overline{X}}\rangle\ |\forall x \in X\}$ is called the value of fuzzy rough of x in A, still written as x.

Suppose X is a fuzzy rough set in A, when A is a finite set, then

$$X = \sum_{i=1}^{n} \langle x, \mu_{\underline{X}}(x_i),\ \mu_{\overline{X}}(x_i)\rangle\ |x_i, x_i \in \overline{X}.$$

When A is continuous, then

$$X = \int \langle \mu_{\underline{X}},\ \mu_{\overline{X}}\rangle\ |x, \forall x \in \overline{X}.\ over\ A.$$

The whole fuzzy rough set in A is renowned by $F^R(X)$.

Let $X = (\underline{X},\ \overline{X})$ be a fuzzy rough set in A, $X^c = \left(\underline{X}^c, \overline{X}^c\right)$ is called the complementary set of $X = (\underline{X},\ \overline{X})$, where $\mu_{\underline{X}^c}(x) = 1 - \mu_{\underline{X}}(x),\ \forall x \in \underline{X};\ \ \mu_{\overline{X}^c}(x) = 1 - \mu_{\overline{X}}(x),\ \forall x \in \overline{X}$.

The order relation in X is defined by the following condition:

$$x \leq y \Leftrightarrow \mu_{\underline{X}}(x) \leq \mu_{\underline{X}}(y)\ and\ \mu_{\overline{X}}(x) \leq \mu_{\overline{X}}(y).$$

Similarity Measures: A similarity measure or similarity function is a real-valued function that enumerates the similarity between two objects. Although no specific definition of a similarity measures subsisted, usually such measures are some implication of the inverse of distance measures. Similarity measures are exploited in system configuration. Higher scores are given to more-similar quality, and lower or negative scores for dissimilar quality.

A similarity measure is an authoritative measure if it convinced following condition:

1. $0 \leq S(A, B) \leq 1$
2. $S(A, B) = 1$ (or maximum similarity) if and only if $A = B$.
3. $S(A, B) = 0$ if and only if $B = 1 - A\ or\ (A^c)$.
4. $S(A, B) = S(B, A)$ for all A and B, where $S(A, B)$ is the similarity between data objects A and B.
5. If $A \subseteq B \subseteq C$, then $S(A, B) \geq S(A, C)$, and $S(B, C) \geq S(A, C)$.

3. Some Similarity Measures

In recent years, various researchers and authors recommended and researched different similarity measure of fuzzy sets and Intuitionistic fuzzy sets [14-19]. Chen [18] is one that gave the similarity measure related to intuitionistic fuzzy set for measuring the degree of similarity between elements as follows.

Definition. Let $x = [t_A(x), 1 - f_A(x)]$ and $y = [t_A(y), 1 - f_A(y)]$ be two fuzzy values in IFS A. A degree of similarity between the fuzzy values x and y can be estimated by the function M_C,

$$M_C(x, y) = 1 - \frac{1}{2}|S(x) - S(y)|, \qquad (3.1)$$

where $S(x) = t_A(x) - f_A(x)$ and $S(y) = t_A(y) - f_A(y)$.

Definition. Let $x = [\, t_A(x), 1 - f_A(x)]$ and $y = [t_A(y), 1 - f_A(y)]$ be two fuzzy values in IFS A. A degree of similarity between the fuzzy values x and y can be calculated by the function M_H,

$$M_H(x, y) = 1 - \frac{1}{2}(|t_A(x)| - |t_A(y)| + |f_A(x)| - |f_A(y)|) \qquad (3.2)$$

In [16], Zhang provide a similarity measure between two fuzzy rough sets and fuzzy rough values as follows.

Definition. Let A be a fuzzy rough set in $X, x = \langle \underline{\mu_A(x)}, \overline{\mu_A(x)} \rangle, y = \langle \underline{\mu_A(y)}, \overline{\mu_A(y)} \rangle$ the fuzzy rough values in A. The degree of similarity between the fuzzy rough values x and y can be assessed by the function M_Z

$$M_Z(x, y) = 1 - \frac{1}{2}\left(\left|\underline{\mu_A(x)} - \underline{\mu_A(y)}\right| - \left|\overline{\mu_A(x)} - \overline{\mu_A(y)}\right|\right) \qquad (3.3)$$

On the basis of fuzzy information handling he identified some axioms or rule which assured the authenticity of similarity measures of fuzzy rough values. Consider $x = \langle \underline{\mu_A(x)}, \overline{\mu_A(x)} \rangle, y = \langle \underline{\mu_A(y)}, \overline{\mu_A(y)} \rangle$ *and* $z = \langle \underline{\mu_A(z)}, \overline{\mu_A(z)} \rangle$ be the fuzzy rough values in a fuzzy rough set A. Then M is the similarity measure to quantify the degree of similarity between elements in A, if persuade following conditions:

1. (symmetry) $M(x, y) = M(y, x)$.
2. (monotony) if $x \le y \le z$, then
$$M(x, z) \le min\{M(x, y), M(y, z)\}.$$
3. $M(x, y) = 0$ iff $(x = \langle 0, 0 \rangle)$ *and* $(y = \langle 1, 1 \rangle)$; or $(x = \langle 1, 1 \rangle)$ *and* $(y = \langle 0, 0 \rangle)$;
$M(x, y) = 1$ iff $\left(\underline{\mu_A(x)} = \underline{\mu_A(y)}\right)$ *and* $\left(\overline{\mu_A(x)} = \overline{\mu_A(y)}\right)$.
4. $M(x, y) = M(x^c, y^c)$.
5. $\forall x \in X$, if $M(x, y) = M(x, z) \Longrightarrow M(y, z) = 1$.

Here the order relations of the fuzzy rough value as follow:

$$x \le y \Leftrightarrow \left(\underline{\mu_A(x)} \le \underline{\mu_A(y)}\right) \; and \; \left(\overline{\mu_A(x)} \le \overline{\mu_A(y)}\right).$$

4. The Similarity measure between Fuzzy Rough Sets and its Elements

4.1. The measures of Similarity between the Elements of Fuzzy Rough Set:

Definition: let $A \in F^R(X)$ and $x = \langle \underline{\mu_A(x)}, \overline{\mu_A(x)} \rangle \in A$. Then:

1. $\tau_x = \overline{\mu_A(x)} - \underline{\mu_A(x)}$ is called the degree of indeterminacy of the element $x \in A$.
2. $\rho_x = \underline{\mu_A(x)} + \tau_x \underline{\mu_A(x)} = (1 + \tau_x)\underline{\mu_A(x)}$ is called the degree of favor $x \in A$
3. $\sigma_x = 1 - \overline{\mu_A(x)} + \tau_x \left(1 - \overline{\mu_A(x)}\right)$ is called the degree of against $x \in A$.

Remarks. (1) x is more unspecified for superior value of τ_x. If $\tau_x = 1, i.e. \overline{\mu_A(x)} = 1 \; and \; \underline{\mu_A(x)} = 0$, then we know nothing for x; if $\forall x \in A, \tau_x = 0$, then the fuzzy rough set A is a fuzzy set; if $\forall x \in A$, $\overline{\mu_A(x)} = \underline{\mu_A(x)} = 1(0)$, then the fuzzy rough set A is a common set.

(2) The prior knowledge is considered when defining ρ_x and σ_x, it can be interpreted by the voting model. A fuzzy rough value $[0.4, 0.8]$ can be interpreted as "the vote for resolution is 4 in favor, 2 against and 4 abstention". Then $\rho_x = 0.4 + 0.4(0.8 - 0.4) = 0.56$ can be interpreted as "considering the vote for resolution as above, besides 4 in favor, it is possible that there is 0.4×4 favor in the abstention". Alike $\sigma_x = 0.2 + 0.2(0.8 - 0.4) = 0.28$ can be interpreted as "considering the vote for resolution as above, besides 2 against, it is possible that there is 0.2×4 against in the 4 abstention."

In 2008 Qi et al. proposed a similarity measures between fuzzy rough sets and its elements. The similarity measures defined by them as:

Definition: Let $A \in F^R(X)$ and τ_x, ρ_x, σ_x as above. $x = \langle \underline{\mu_A(x)}, \overline{\mu_A(x)} \rangle, y = \langle \underline{\mu_A(y)}, \overline{\mu_A(y)} \rangle$ in A. The similarity degree between x and y can be evaluated by the function M,

$$M(x, y) = 1 - \frac{1}{2}(\rho_{xy} + \sigma_{xy}), \qquad (4.1)$$

where $\rho_{xy} = |\rho_x - \rho_y|$ and $\sigma_{xy} = |\sigma_x - \sigma_y|$.

Corresponding to similarity measure – (1) of the elements of Fuzzy rough set we define trigonometric similarity measures of fuzzy rough sets elements as:

$$M_{sin}(x, y) = \sin\frac{\pi}{2}\left(1 - \frac{1}{2}(\rho_{xy} + \sigma_{xy})\right), \qquad (4.2)$$

We will prove validity of these similarity measures by using following theorem.

Theorem 1. Let $A \in F^R(X)$ and τ_x, ρ_x, σ_x as above. $x = \langle \underline{\mu_A(x)}, \overline{\mu_A(x)} \rangle, y = \langle \underline{\mu_A(y)}, \overline{\mu_A(y)} \rangle$ in A. Then the similarity measure

$$M_{sin}(x, y) = \sin\frac{\pi}{2}\left(1 - \frac{1}{2}(\rho_{xy} + \sigma_{xy})\right)$$

is a valid measure.

Proof: To prove above theorem we first prove following lemma:

Lemma 1.1. Let $A \in F^R(X)$ and $M_{sin}(x, y)$ is defined as above. Since $x = \langle \underline{\mu_A(x)}, \overline{\mu_A(x)} \rangle, y = \langle \underline{\mu_A(y)}, \overline{\mu_A(y)} \rangle$ and $z = \langle \underline{\mu_A(z)}, \overline{\mu_A(z)} \rangle \in A$. If $x \le y \le z$, then $M_{sin}(x, z) \le min\{M_{sin}(x, y), M_{sin}(y, z)\}$.

Proof. If $x \le y$, then $\underline{\mu_A(x)} \le \underline{\mu_A(y)}$ and $\overline{\mu_A(x)} \le \overline{\mu_A(y)}$. Let $\underline{\mu_A(x)} = \underline{\mu_A(y)} - u$ and $\overline{\mu_A(x)} = \overline{\mu_A(y)} - v$, where $u > 0$ and $v > 0$, Then $\tau_x = \tau_y - (v - u)$.

$\rho_y - \rho_x = u(1 + \tau_x) + \underline{\mu_A(y)}(v - u) \ge u\underline{\mu_A(y)} + \underline{\mu_A(y)}(v - u) = v\underline{\mu_A(y)} \ge 0$, so $\rho_z - \rho_x = \rho_z - \rho_y + \rho_y - \rho_x \ge \max\{\rho_z - \rho_y, \rho_y - \rho_x\}$, by the non-negativity of $\rho_z - \rho_y$ and $\rho_y - \rho_x$, we have $\rho_{xz} \ge max\{\rho_{xy}, \rho_{yz}\}$, then $\left(1 - \frac{1}{2}(\rho_{xz} + \sigma_{xz})\right) \le min\left\{\left(1 - \frac{1}{2}(\rho_{xy} + \sigma_{xy})\right), \left(1 - \frac{1}{2}(\rho_{yz} + \sigma_{yz})\right)\right\}$.

Thus we have $M_{sin}(x, z) \le min\{M_{sin}(x, y), M_{sin}(y, z)\}$.

Lemma 1.2. Let $A \in F^R(X)$ and $M_{sin}(x, y)$ as above. Then:

1. $M_{sin}(x, y) = 0 \Leftrightarrow (x = \langle 0, 0 \rangle \; and \; y = \langle 1, 1 \rangle) \; or \; (x = \langle 1, 1 \rangle \; and \; y = \langle 0, 0 \rangle)$.
2. $M_{sin}(x, y) = 1 \Leftrightarrow \underline{\mu_A(x)} = \underline{\mu_A(y)} \; and \; \overline{\mu_A(x)} = \overline{\mu_A(y)}$.

Proof. (1) Initially, since $0 \le \rho_x = \underline{\mu_A(x)} + \tau_x\underline{\mu_A(x)} = \underline{\mu_A(x)} + \left(\overline{\mu_A(x)} - \underline{\mu_A(x)}\right)\underline{\mu_A(x)} \le \underline{\mu_A(x)} + \left(\overline{\mu_A(x)} - \underline{\mu_A(x)}\right) = \overline{\mu_A(x)} \le 1$, then $\rho_{xy} = |\rho_x - \rho_y| \le 1$. Correspondingly, we have $\sigma_{xy} \le 1$.

Subsequently, $M_{sin}(x, y) = 0 \Leftrightarrow \rho_{xy} + \sigma_{xy} = 2 \Leftrightarrow \rho_{xy} = 1 = \sigma_{xy}$.

If $\sigma_{xy} = |\sigma_x - \sigma_y| = 1$, then from $0 \leq \sigma_x \leq 1$ and $0 \leq \sigma_y \leq 1$ implies $\sigma_x = 1$ and $\sigma_y = 0$ or $\sigma_x = 0$ and $\sigma_y = 1$. Similarly, if $\rho_{xy} = |\rho_x - \rho_y| = 1$, we get $\rho_x = 1$ and $\rho_y = 0$ or $\rho_x = 0$ and $\rho_y = 1$.

When $\sigma_x = 1$, because $\sigma_x = 1 - \overline{\mu_A(x)} + \tau_x\left(1 - \overline{\mu_A(x)}\right) \leq 1 - \overline{\mu_A(x)} + \tau_x \leq 1 - \underline{\mu_A}(x)$, thus $\underline{\mu_A}(x) = 0$; Since $0 \leq \overline{\mu_A(x)} \leq 1$, so $1 = \sigma_x = \left(1 - \overline{\mu_A(x)}\right)\left(1 + \overline{\mu_A(x)}\right) = 1 - \left(\overline{\mu_A(x)}\right)^2$, it implies that $\overline{\mu_A(x)} = 0$. When $\sigma_y = 0$, we have $\overline{\mu_A(y)} = 1$ by non-negativity of $\overline{\mu_A(y)}$ and τ_y. So, we have $\overline{\mu_A(x)} = \underline{\mu_A}(x) = \rho_x = 0$ and $\overline{\mu_A(y)} = 1$ when $\sigma_x = 1$ and $\sigma_y = 0$.

Similarly, we have $\overline{\mu_A(y)} = \underline{\mu_A}(y) = \rho_y = 0$ and $\overline{\mu_A(x)} = 1$ when $\sigma_y = 1$ and $\sigma_x = 0$.

Similarly, we retrieve that $\underline{\mu_A}(y) = 0$ and $\underline{\mu_A}(x) = 1$, when $\rho_y = 1$ and $\rho_x = 0$; and $\rho_x = 1$ and $\rho_y = 0$ in that order.

Thus, $\rho_{xy} = \sigma_{xy} = 1 \Leftrightarrow (x = \langle 0,0\rangle \; and \; y = \langle 1,1\rangle) \; or \; (x = \langle 1,1\rangle \; and \; y = \langle 0,0\rangle)$.

(2) First of all, $M_{sin}(x,y) = 1 \Leftrightarrow \rho_{xy} = \sigma_{xy} = 0 \Leftrightarrow \rho_x = \rho_y \; and \; \sigma_x = \sigma_y \Leftrightarrow \left(\underline{\mu_A}(x) + 1 - \overline{\mu_A(x)}\right)(1 + \tau_x) = \left(\underline{\mu_A}(y) + 1 - \overline{\mu_A(y)}\right)(1 + \tau_y)$ and $\left(\underline{\mu_A}(x) - 1 + \overline{\mu_A(x)}\right)(1 + \tau_x) = \left(\underline{\mu_A}(y) - 1 + \overline{\mu_A(y)}\right)(1 + \tau_y) \Rightarrow (1 + \tau_y)(1 - \tau_y) = (1 + \tau_x)(1 - \tau_x) \Rightarrow \tau_x = \tau_y \Rightarrow \overline{\mu_A(x)} - \underline{\mu_A}(x) = \overline{\mu_A(y)} - \underline{\mu_A}(y)$.

Second, from $\tau_x = \tau_y$ and $\left(\underline{\mu_A}(x) - 1 + \overline{\mu_A(x)}\right)(1 + \tau_x) = \left(\underline{\mu_A}(y) - 1 + \overline{\mu_A(y)}\right)(1 + \tau_y)$ implies $\overline{\mu_A(x)} + \underline{\mu_A}(x) = \overline{\mu_A(y)} + \underline{\mu_A}(y)$, thus we have $\underline{\mu_A}(x) = \underline{\mu_A}(y)$ and $\overline{\mu_A(x)} = \overline{\mu_A(y)}$. It is clear that $\rho_{xy} = \sigma_{xy} = 0$ if $\underline{\mu_A}(x) = \underline{\mu_A}(y)$ and $\overline{\mu_A(x)} = \overline{\mu_A(y)}$. Thus $M_{sin}(x,y) = 1$.

Lemma 1.3. Let $A \in F^R(X)$ and $M_{sin}(x,y)$ as above. Then $M_{sin}(x,y) = M_{sin}(y,x)$ and $M_{sin}(x,y) = M_{sin}(x^c,y^c)$.

Lemma 1.4. Let $A \in F^R(X)$ and $M_{sin}(x,y)$ as above. If $M_{sin}(x,y) = M_{sin}(x,z)\; \forall x \in X$, then $M_{sin}(y,z) = 1$.

Proof. If for all $x \in X$, $M_{sin}(x,y) = M_{sin}(x,z)$ then $(\rho_{xy} + \sigma_{xy}) = (\rho_{xz} + \sigma_{xz})$ and thus $\left(|\rho_x - \rho_y| + |\sigma_x - \sigma_y|\right) = (|\rho_x - \rho_z| + |\sigma_x - \sigma_z|)$. Let $x \in X$ and x satisfies $\sigma_x = 1$, by the proof of above lemma, we have $\rho_x = 0$, thus $\left(\rho_y + |1 - \sigma_y|\right) = (\rho_z + |1 - \sigma_z|)$. Then $(\rho_y - \sigma_y) = (\rho_z - \sigma_z)$ this implies $\rho_{yz} = \sigma_{yz}$. Let $x \in X$ such that $\underline{\mu_A}(x) = 0$, $\overline{\mu_A(x)} = 1$, then $\rho_x = \sigma_x = 0$, it implies that $(\rho_y + \sigma_y) = (\rho_z + \sigma_z)$. Thus $\rho_y = \rho_z$ and $\sigma_y = \sigma_z$ then $\rho_{yz} = \sigma_{yz} = 0$, so $M_{sin}(y,z) = 1$.

Since above all four lemmas satisfies the property of validity for a similarity measure of the fuzzy rough sets elements. Thus our proposed measure is a valid measure.

4.2. The Measures of Similarity between the Fuzzy Rough Sets:

Definition: Let $A, B \in F^R(X)$, $X = \{x_1, x_2, \ldots \ldots, x_m\}$. If $F_A^R(x) = \langle \underline{\mu_A}(x), \overline{\mu_A(x)}\rangle$ is the fuzzy rough value of x in A and $F_B^R(x) = \langle \underline{\mu_B}(x), \overline{\mu_B(x)}\rangle$ is the fuzzy rough value of x in B. Then the degree of similarity between the fuzzy rough sets A and B can be calculated by the following:

$$S_{sin}(A,B) = \frac{1}{m}\sum_{j=1}^{m} M_{sin}\left(F_A^R(x_j), F_B^R(x_j)\right)$$

$$= \frac{1}{m}\sum_{j=1}^{m} \sin\frac{\pi}{2}\left(1 - \frac{\left(\rho_A(x_j) - \rho_B(x_j)\right)}{2} - \frac{\left(\sigma_A(x_j) - \sigma_B(x_j)\right)}{2}\right) \qquad (4.3)$$

where $\rho_A(x_j) = \underline{\mu_A(x_j)} + \left(\overline{\mu_A(x_j)} - \underline{\mu_A(x_j)}\right)\underline{\mu_A(x_j)}$ and $\sigma_A(x_j) = 1 - \overline{\mu_A(x_j)} + \left(\overline{\mu_A(x_j)} - \underline{\mu_A(x_j)}\right)\left(1 - \overline{\mu_A(x_j)}\right)$.

It is clear that $S_{sin}(A, B) \in [0, 1]$, and A and B are more similar for superior value of $S_{sin}(A, B)$.
The following inferences are evident.

Proposition 1. $S_{sin}(A, B) = S_{sin}(B, A),\ S_{sin}(A, B) = S_{sin}(A^c, B^c)$.

Proposition2. $S_{sin}(A, B) = 0 \Leftrightarrow \left(A = \sum_{j=1}^{m}\langle 0, 0\rangle/x_j \text{ and } B = \sum_{j=1}^{m}\langle 1, 1\rangle/x_j\right)$ or $\left(A = \sum_{j=1}^{m}\langle 1, 1\rangle/x_j \text{ and } B = \sum_{j=1}^{m}\langle 0, 0\rangle/x_j\right)$.

Proposition 3. $S_{sin}(A, B) = 1 \Leftrightarrow \overline{\mu_A(x_j)} = \overline{\mu_B(x_j)} \text{ and } \underline{\mu_A(x_j)} = \underline{\mu_B(x_j)}, \forall x_j \in X$.

We may define the order relation between the fuzzy rough sets:

$$A \subseteq B \Leftrightarrow \underline{\mu_A(x)} \le \underline{\mu_B(x)} \text{ and } \overline{\mu_A(x)} \le \overline{\mu_B(x)}, \forall x_j \in X$$

Proposition 4. For all $A, B, C \in F^R(X),\ A \subseteq B \subseteq C \Longrightarrow S_{sin}(A, C) \le min\{S_{sin}(A, B), S_{sin}(B, C)\}$

Since above sine trigonometric measure satisfy all the condition of similarity measure so this measure is a valid measure.

4.3. The Weighting Measures of Similarity between the Fuzzy Rough Sets

Let $A, B \in F^R(X), X = \{x_1, x_2, \ldots \ldots, x_m\}$, where

$$A = \sum_{j=1}^{m}\left[\underline{\mu_A(x_j)}, \overline{\mu_A(x_j)}\right]/x_j, \quad B = \sum_{j=1}^{m}\left[\underline{\mu_B(x_j)}, \overline{\mu_B(x_j)}\right]/x_j, \qquad x_j \in X.$$

Assume that the weight of the element x_j is w_j, where $w_j \in [0, 1]$ and $1 \le j \le m$, then the degree of similarity between the fuzzy rough sets A and B can be calculated by the weighting function as follows:

$$W_{sin}(A, B) = \frac{\sum_{j=1}^{m} w_j M_{sin}\left(F_A^R(x_j), F_B^R(x_j)\right)}{\sum_{j=1}^{m} w_j}$$

$$W_{sin}(A, B) = \frac{\sum_{j=1}^{m} w_j \sin\frac{\pi}{2}\left(1 - \frac{\left(\rho_A(x_j) - \rho_B(x_j)\right)}{2} - \frac{\left(\sigma_A(x_j) - \sigma_B(x_j)\right)}{2}\right)}{\sum_{j=1}^{m} w_j} \quad (4.4)$$

where $W_{sin}(A, B) \in [0, 1]$. The largest value of $W_{sin}(A, B)$ implies more similarity between A and B.

5. Application

The measures of similarity between the FRSs, can be utilized to quantify the significance of an attribute in a specified organized task. Here we analysis the same application which is analyzed by Chengyi et al [16]. Here we demonstrate this dilemma in the environment of colorectal cancer diagnosis. In this case, we analyze the connection between the main prognostic factors and the effects of the patients that are enduring the follow-up course of the colorectal cancer.

We found these types of diseases normally in the developed countries. The colorectal cancer develops primarily in the mucosa lining of bowel. In generally circumstances, the initial movement towards the development of a colorectal cancer is the arrival of polyps. When the abnormal cells within the polyps commence to expand and infect with normal tissue, polyps develop into cancer growths. If no suitable remedy is espoused, then the cancer can extend to the skin and the essential tissues of the bowel wall, and ultimately the cancer may reach to the distant sites like liver. Well, recurrence, metastasis and both (i.e., recurrence and metastasis simultaneously), are promising effects to state the situation of the patient. The major dealing to the colorectal cancer is the surgical ejection of the tumor, while the survival of a patient with the colorectal cancer is reliant on four basic features: (a) the biology of that specific's malignancy, (b) the immune response to the tumor, (c) the time for the cancer patient's life history when the diagnosis is through, and (d) the competence of the treatment. About 50% patients ultimately die from the local recurrence and / or distant metastasis within 5 years after the remedial resection. Therefore, to amplify the possibility of survival, it is crucial to distinguish or expect the recurrent or metastasis tumor in the follow-up so that the suitable treatment is imposed.

The following 16 aspects are considered: liver metastasis, peritoneal metastasis, CT scan for liver, regional lymph node, apical node status, apical node number, adjacent structure invasion, venous invasion, perineural invasion, differentiation, ascites, carcinoembryonic antigen, and surgeon rank, site of tumor, preoperative blood transfusion, and fix to adjacent structure. The patient, who is in the follow-up program, may collapse into any of the following conditions: Metastasis, recurrence, bad and well. If the position of an individual patient can be perfectly determined, then the situation information can be exploited to select a proper therapy. A surgeon can instinctively conclude the belongingness of each patient in the output classes.

Let A be a feature set of a healthy person, $w_j \in [1, 16]$ is the weight of the feature, B is an feature set of a patient with colorectal cancer, if

$$\begin{cases} W_{sin}(A,B) < \alpha_1, & B \text{ is well}, \\ \alpha_1 \le W_{sin}(A,B) < \alpha_2, & B \text{ is recurrence}, \\ \alpha_2 \le W_{sin}(A,B) < \alpha_3, & B \text{ is metastasis}, \\ W_{sin}(A,B) \ge \alpha_3, & B \text{ is bad}, \end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3$ are the threshold values.

Using the above procedure, we have effort to compute the significance of the prognostic factors.

Now we compare our proposed measure with existing measures, for this we consider same numerical example which is considered by the Qi et al [17].

Let A be a characteristic set of a patient, for the suitability of discussion, we adopt the main 5 characteristics and enumerate the characteristic, respectively represented a, b, c, d, e; as these characteristics generally are language variable, for every characteristic, we set up FRSs function by fuzzy method and obtain their attributes values. Let mode $A = \{[0.3, 0.5], [0.4, 0.6], [0.6, 0.8], [0.5, 0.9], [0.9, 1]\}$, A_1, A_2, A_3 and A_4 are the attribute sets of the samples indicated metastasis, recurrence, bad and well shown in Table 1.

	A	b	C	D	e
A_1	[0.4, 0.6]	[0.3, 0.7]	[0.5, 0.9]	[0.5, 0.8]	[0.6, 0.8]
A_2	[0.2, 0.4]	[0.3, 0.5]	[0.2, 0.3]	[0.7, 0.9]	[0.8, 1]
A_3	[0.1, 0.1]	[0, 0]	[0.2, 0.3]	[0.1, 0.2]	[0.6, 0.6]
A_4	[0.8, 0.8]	[0.9, 1]	[1, 1]	[0.7, 0.8]	[0.6, 0.6]

Table 1: Attribute sets of the sample

Using the method by Chengyi et al [16], we calculate the similarity measures between the patient A and the sample A_1, A_2, A_3 and A_4: $A = \{[0.3, 0.5], [0.4, 0.6], [0.6, 0.8], [0.5, 0.9], [0.9, 1]\}$,

$$S_{Ch}(A, A_1) = 1 - \frac{1}{10}\begin{pmatrix} |0.3-0.4| + |0.5-0.6| + |0.4-0.3| + |0.6-0.7| + |0.6-0.5| \\ + [0.8-0.9] + |0.5-0.5| + |0.9-0.8| + |0.9-0.6| + |1-0.8| \end{pmatrix} = 0.88$$

Similarly we calculate other similarity measure between the patient A and other sample as:

$$S_{Ch}(A, A_2) = 0.88; \quad S_{Ch}(A, A_3) = 0.49; \quad S_{Ch}(A, A_4) = 0.67.$$

Since $S_{Ch}(A, A_1) = S_{Ch}(A, A_2)$, we could not tell A is a metastasis patient or a recurrence patient. It shows that the method defined by Chengyi et al [16] could not distinguish A is similar to A_1 or A_2. While using the measure proposed by Qi et al [17], A could be distinguished correctly, so he may be cured suitably. The process is followed:

	a	b	C	D	E
$\underline{\mu_{A_1}}$	0.40	0.30	0.50	0.50	0.60
$\overline{\mu_{A_1}}$	0.60	0.70	0.90	0.80	0.80
τ_{A_1}	0.20	0.40	0.40	0.30	0.20
ρ_{A_1}	0.48	0.42	0.70	0.65	0.72
σ_{A_1}	0.48	0.42	0.14	0.26	0.24

	a	b	c	D	e
$\underline{\mu_{A_2}}$	0.20	0.30	0.20	0.70	0.80
$\overline{\mu_{A_2}}$	0.40	0.50	0.70	0.90	1.00
τ_{A_2}	0.20	0.20	0.50	0.20	0.20
ρ_{A_2}	0.24	0.36	0.30	0.84	0.96
σ_{A_2}	0.72	0.60	0.45	0.12	0.00

	a	b	C	D	E
$\underline{\mu_{A_3}}$	0.10	0.00	0.20	0.10	0.20
$\overline{\mu_{A_3}}$	0.10	0.00	0.30	0.20	0.20
τ_{A_3}	0.00	0.00	0.10	0.10	0.00
ρ_{A_3}	0.10	0.00	0.22	0.11	0.20
σ_{A_3}	0.30	1.00	0.77	0.88	0.80

	a	b	c	D	e
$\underline{\mu_{A_4}}$	0.80	0.90	1.00	0.70	0.60
$\overline{\mu_{A_4}}$	0.80	1.00	1.00	0.80	0.60
τ_{A_4}	0.00	0.10	0.00	0.10	0.00
ρ_{A_4}	0.80	0.99	1.00	0.77	0.60
σ_{A_4}	0.20	0.00	0.00	0.22	0.40

	a	B	c	d	E
$\underline{\mu_A}$	0.30	0.40	0.60	0.50	0.90
$\overline{\mu_A}$	0.50	0.60	0.80	0.90	1.00
τ_A	0.20	0.20	0.20	0.40	0.10
ρ_A	0.36	0.48	0.72	0.70	0.99
σ_A	0.60	0.48	0.24	0.14	0.00

$$S_{Qi}(A, A_1) = 1 - \frac{1}{10}\begin{pmatrix} |0.36 - 0.48| + |0.6 - 0.48| + |0.48 - 0.42| + |0.48 - 0.42| + |0.72 - 0.70| \\ + [0.24 - 0.14] + |0.70 - 0.65| + |0.14 - 0.26| + |0.99 - 0.72| + |0.00 - 0.24| \end{pmatrix} = 0.884$$

Similarly we calculate other similarity measure between the patient A and other sample as:

$$S_{Qi}(A, A_2) = 0.87; \quad S_{Qi}(A, A_3) = 0.449; \quad S_{Qi}(A, A_4) = 0.671.$$

Basing the above result, mode A is supposed to a metastasis patient.

Now we used our proposed measures, the process as follows:

$$
\begin{aligned}
S_{sin}(A, A_1) &= \frac{1}{5}\left[\sin\frac{\pi}{2}\left(1-\frac{1}{2}(|0.36-0.48|+|0.6-0.48|)\right)\right. \\
&\quad + \sin\frac{\pi}{2}\left(1-\frac{1}{2}(|0.48-0.42|+|0.48-0.42|)\right) \\
&\quad + \sin\frac{\pi}{2}\left(1-\frac{1}{2}(|0.72-0.70|+|0.24-0.14|)\right) \\
&\quad + \sin\frac{\pi}{2}\left(1-\frac{1}{2}(|0.70-0.65|+|0.14-0.26|)\right) \\
&\quad \left.+ \sin\frac{\pi}{2}\left(1-\frac{1}{2}(|0.99-0.72|+|0.00-0.24|)\right)\right] \\
&= 0.977
\end{aligned}
$$

Similarly we calculate other similarity measure between the patient A and other sample as:

$$S_{sin}(A, A_2) = 0.967; \quad S_{sin}(A, A_3) = 0.624; \quad S_{sin}(A, A_4) = 0.845.$$

Here we also observe that our proposed measure distinguished A correctly, so he may be treated properly. Basing the above result, mode A is supposed to a metastasis patient.

6. Conclusions

As we know that fuzzy set and rough set are complementary in nature, but their hybridization is possible which is significant in study of vague concept related problems. Here we discussed such hybridization known as fuzzy rough set. The fuzzy rough set is a contemporary theory characterized imprecise information. Measures of similarity between fuzzy rough sets, as an imperative subject in fuzzy mathematics, have achieved interest in researchers for their open application of real world. Persistently glancing for the improved similarity measure method is a tracking of fuzzy set and rough set theory. However, none of the any similarity measure methods are all-significant, and all have the restricted of its usage. In this paper we defined a similarity measure for measuring the degree of similarity between elements and between fuzzy rough sets. Ultimately, a promising application of the similarity measure is indicated and comparison is also discussed with exist one measures, which shows that our measure is better than existing one. As the intended similarity measures have several proficient properties it can provide an effective approach to measuring the similarity between fuzzy rough sets. In choruses, it will be a very significant research subject that exploring the invariance properties of pattern recognition of the various similarity measure methods.

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