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MORE RESULTS ON COMPLEMENTARY TREE NIL DOMINATION NUMBER OF A GRAPH

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Abstract:
A set D of a graph G = (V, E) is a dominating set, if every vertex in V – D is adjacent to some vertex in D. The domination number γ(G) of G is the minimum cardinality of a dominating set. A dominating set D of a connected graph G is called a complementary tree nil dominating set if the induced sub graph <V – D> is a tree and V – D is not a dominating set. The minimum cardinality of a complementary tree nil dominating set is called the complementary tree nil domination number of G and is denoted by $\gamma_{ctnd}(G)$. In this paper, bounds for $\gamma_{ctnd}(G)$ and its exact values for some particular classes of graphs are found. Some more results on complementary tree nil domination number are also established.

***Key words**: complementary tree domination number, complementary trees nil domination number.*

1. Introduction

Graphs discussed in this paper are finite, undirected and simple graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. For v ∈ V(G), the neighborhood N(v) of v is the set of all vertices adjacent to v in G. N[v] = N(v) ∪ {v} is called the closed neighborhood of v. A vertex v ∈ V(G) is called a support if it is adjacent to a pendant vertex. (That is, a vertex of degree one). If x = uv is a line of G, and w is not a point of G, then x is subdivided when it is replaced by the lines uw and wv. If every line of G is subdivided, the resulting graph is the subdivision graph S(G). The concept of domination in graphs was introduced by Ore[5]. A set D ⊆ V(G) is said to be a dominating set of G, if every vertex in V(G) - D is adjacent to some vertex in D. A minimum dominating set in a graph G is a dominating set of minimum cardinality. The cardinality of a minimum dominating set in G is called the domination number of G and is denoted by γ(G). The concept of domatic partition in graphs was introduced by E.J. Cockayne and S.T. Hedetniemi[2]. A partition Δ of its vertex set V(G) is called a domatic partition of G if each class of Δ is a dominating set in G. The maximum number of classes of a domatic partition of G is called the domatic number of G and is denoted by d(G). A dominating set D of a connected graph G is a nonsplit dominating set, if the induced subgraph<V(G) - D> is connected. Muthammai, Bhanumathi and Vidhya[4] introduced the concept of complement tree dominating set. A dominating set D⊆ V(G) is said to be complementary tree dominating set (ctd-set) if the induced sub graph <V(G) - D> is a tree. The minimum cardinality of a ctd-set is called the complementary tree domination number of G and is denoted by $\gamma_{cnd}(G)$. Any undefined terms in this paper may be found in Harary[1]. Here, G is a connected graph with p vertices and q edges.

We introduced the concept of complementary tree nil dominating set in [3]. A dominating set D ⊆V(G) is said to be a complementary tree nil dominating set (ctnd-set) if the induced subgraph < V(G) - D >is a tree and V(G) - D is not a dominating set. The minimum cardinality of a ctnd-set is called the complementary tree nil domination number of G and is denoted by $\gamma_{ctnd}(G)$.

In this paper, bounds for $\gamma_{ctnd}(G)$ and its exact values for some particular classes of graphs and also relationships between complementary tree nil domination number and other related parameters are found.

2. Prior Results

Theorem 2.1.[2] For any connected graph G, d(G) ≤ $\delta(G) + 1$.

Theorem 2.2.[3] For any connected graph G with p vertices, $2 \leq \gamma_{ctnd}(G) \leq p$, where $p \geq 2$.

Theorem 2.3.[3]For any connected graph G, $\delta(G) + 1 \leq \gamma_{ctnd}(G)$.

Theorem 2.4.[3]Let G be a connected graph with p vertices. Then $\gamma_{ctnd}(G) = 2$ if and only if G is a graph obtained by attaching a pendant edge at a vertex of degree p - 2 in $T + K_1$, where T is a tree on (p – 2) vertices.

Theorem 2.5.[3] For any connected graph G, $\gamma_{ctnd}(G) = p$ if and only if $G \cong K_p$, where $p \geq 2$.

Theorem 2.6.[3] Let G be a connected graph with $p \geq 3$ and $\delta(G) = 1$. Then $\gamma_{ctnd}(G) = p - 1$ if and only if the subgraph of G induced by vertices of degree atleast 2 is K_2 or K_1.

That is, G is one of the graphs $K_{1, p-1}$ or $S_{m,n}$($m + n = p$, $m, n \geq 1$), where $S_{m, n}$ is a bistar which is obtained by attaching m-1 pendant edges at one vertex of K_2 and n-1 pendant edges at other vertex of K_2.

Theorem 2.7.[3] Let G be a connected noncomplete graph with$\delta(G) \geq 2$. Then $\gamma_{ctnd}(G) = p - 1$ if and only if each edge of G is a dominating edge.

Theorem 2.8.[3] Let T be a tree on p vertices such that $\gamma_{ctnd}(T) \leq p - 2$. Then $\gamma_{ctnd}(T) = p - 2$ if and only if T is one of the following graphs.

(i) T is obtained from a path P_n ($n \geq 4$ and $n < p$) by attaching pendant edges at atleast one of the end vertices of P_n.

(ii) T is obtained from P_3 by attaching pendant edges either at both the end vertices or at all the vertices of P_3.

Notation 2.9.[3] Let $\mathcal{G}$ be the class of connected graphs G with $\delta(G) = 1$ having one of the following properties.

(a) There exist two adjacent vertices u, v in G such that $\deg_G(u) = 1$ and <V(G) - {u, v}> contains P_3 as an induced subgraph such that end vertices of P_3 have degree atleast 2 and the central vertex of P_3 has degree atleast 3.

(b) Let P be the set of all pendant vertices in G and let there exist a vertex $v \in V(G) - P$ having minimum degree in V(G) - P and is not a support of G such that V(G) – $(N_{V-P}[v] - P)$ contains P_3 as an induced subgraph such that the end vertices of P_3 have degree atleast 2 and the central vertex of P_3 has degree atleast 3.

Theorem 2.10.[3] Let G be a connected graph with $\delta(G) = 1$ and $\gamma_{ctnd}(G) \neq p - 1$. Then $\gamma_{ctnd}(G) = p - 2$ if and only if G does not belong to the class $\mathcal{G}$ of graphs.

Theorem 2.11.[3] Let G be a connected, noncomplete graph with p vertices ($p \geq 4$) and $\delta(G) \geq 2$. Then $\gamma_{ctnd}(G) = p - 2$ if and only if G is one of the following graphs.

(a) A cycle on atleast five vertices.

(b) A wheel with six vertices (W_6).

(c) G is the one point union of complete graphs.

(d) G is obtained by joining two complete graphs by edges.

(e) G is a graph such that there exists a vertex v ∈ V(G) such that G – v is a complete graph on (p – 1) vertices .

(f) G is a graph such that there exists a vertex v ∈ V(G) such that G – v is $K_{p-1} - e$, (e ∈ E(K_{p-1})) and N(v) contains atleast one vertex of degree (p – 3) in $K_{p-1} - e$.

Theorem 2.12. [4] $\gamma_{ctd}(T)$ = m if and only if each vertex of degree atleast 2 is a support, where m is the number of pendent vertices in T.

3. Main Results

Definition 3.1.

The one point union $C_n^{(t)}$ of t-copies of cycle C_n is the graph obtained by taking a new vertex u as a common vertex such that any two distinct cycles C_i and C_j are edge disjoint and do not have any vertex in common except u.

Theorem 3.1.

For t≥ 2 and n ≥ 4, $\gamma_{ctnd}(C_n^{(t)}) = \begin{cases} t+2, & \text{if } n = 4 \\ (n-3)t+1, & \text{if } n \geq 5. \end{cases}$

Proof.

Let G = $C_n^{(t)}$ and u be the vertex of union of t cycles of length n. G has t(n – 1) + 1 vertices. Let the vertex set of k[th] cycle in G be $V_k = \{u, v_1^k, v_2^k, \ldots, v_{n-1}^k\}$ and edge set be $E_k = \{(u, v_1^k), (u, v_{n-1}^k)\} \cup \bigcup_{i=1}^{n-2}(v_i^k, v_{i+1}^k)$. Therefore, $V(G) = \bigcup_{k=1}^{t} V_k$, $E(G) = \bigcup_{k=1}^{t} E_k$.

Case 1: n = 4 and |V(G)| = 3t + 1.

If $D = \bigcup_{k=1}^{t}\{v_2^k\} \cup \{v_1^1, v_3^1\} \subseteq V(G)$, then D is a dominating set of G. Also $\langle V(G) - D \rangle \cong K_{1,2t-2}$, $N(v_2^1) \subseteq D$ and hence D is a ctnd-set of G. Therefore, $\gamma_{ctnd}(C_4^{(t)}) \leq |D|$ = t + 2. Let D' be any ctnd- set of $C_4^{(t)}$. Since D' contains atleast one vertex v such that $N(v) \subseteq D'$, D' contains vertices of $N(v_i)$ for some v_i, $v_i \in V(G)$, where deg(v_i) = 2 in G. To dominate vertices which are adjacent to u, a vertex from each cycle must belong to D' except one vertex, and hence D' contains atleast 3 + t – 1 vertices. Therefore, $|D'| \geq t + 2$.

That is, $\gamma_{ctnd}(C_4^{(t)}) = t + 2$.

Case 2: n ≥ 5 and |V(G)| = (n-1) t + 1.

Let $D = \bigcup_{k=1}^{t}\{v_2^k, v_3^k, \ldots, v_{n-2}^k\} \cup \{v_1^1\} \subseteq V(G)$. Then $\langle V(G) - D \rangle \cong K_{1,2t-1}$, $N(v_i^k) \subseteq D$, i = 3, … n-1, k = 1, 2, …, t and hence D is a ctnd-set of G. Therefore, $\gamma_{ctnd}(G) = |D| \leq t(n-3) + 1$.

Since $\gamma_{ctnd}(C_n)$ = n – 2, n ≥5 and there are t cycles and one vertex is common to all cycles, $\gamma_{ctnd}(C_n^{(t)}) \geq t(n-2) - (t-1) \geq t(n-3) + 1$.

Hence $\gamma_{ctnd}(C_n^{(t)}) = t(n-3) + 1$.

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Example 3.1.

For the graph $C_6^{(3)}$ given in Figure 3.1, D = $\bigcup_{k=1}^{3}\{v_2^k, v_3^k, v_4^k\} \cup \{v_1^1\}$ is a minimum ctnd-set of $C_6^{(3)}$ and hence $\gamma_{ctnd}(C_6^{(3)}) = |D| = 10$.

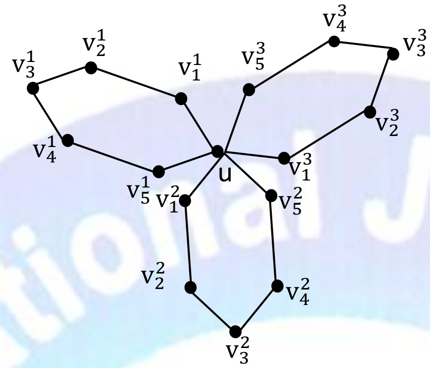

Figure 3.1

Definition 3.2.

Let $G_1, G_2, \ldots, G_k$ be k copies of a graph G, where $k \geq 2$. G(k) is a graph obtained by adding an edge from G_i to G_{i+1}; i = 1, 2, 3, …, k-1 and the graph G(k) is called the path union of k copies of the graph G.

Theorem 3.2.

Let $C_n(t)$, $t \geq 2$, $n \geq 3$ be the path union of t cycles of length n. Then

$$\gamma_{ctnd}(C_n(t)) = \begin{cases} t+2, & \text{if } n = 3 \\ 2t+1, & \text{if } n = 4 \\ (n-2)t, & \text{if } n \geq 5 \end{cases}$$

Proof.

Let $C_n(t)$ denote the path union of t copies of the cycles C_n with vertices v_i^j in the j^{th} copy of C_n, where $1 \leq i \leq n$ and $1 \leq j \leq t$. Let the vertices v_1^k and v_1^{k+1} $(1 \leq k \leq t-1)$ be joined by an edge. The vertices v_1^1 and v_1^t are of degree 3, the vertices v_1^k, $2 \leq k \leq t-1$ are of degree 4 and the remaining vertices are of degree 2 and $C_n(t)$ has nt vertices.

Case 1: n = 3

Here $|V(C_3(t))| = 3t$. Let D = $\bigcup_{i=1}^{t}\{v_2^i\} \cup \{v_1^1, v_3^1\}$. Then D is a dominating set of $C_3(t)$. Also $\langle V(C_3(t)) - D \rangle \cong 2t-2$, $N(v_2^1) \subseteq D$ and hence D is a minimum ctnd-set of $C_3(t)$.

Therefore, $\gamma_{ctnd}(C_3(t)) = |D| = t+2$.

Case 2: n = 4

Here $|V(C_4(t))|$ = 4t. Let $D = \cup_{i=1}^{t}\{v_2^i, v_3^i\} \cup \{v_4^1\}$. Then D is a dominating set of G. Also $<V(C_4(t)) - D>$ is a tree obtained by attaching a pendant edge at a vertex of P_{t-1} of $P_{t-1} \circ K_1$, $N(v_3^1) \subseteq D$ and D is a ctnd-set of G and hence, $\gamma_{ctnd}(C_4(t)) = |D| \leq 2t+1$. Let D' be any ctnd- set of G. Since D' contains atleast one vertex v such that $N(v) \subseteq D'$, D' contains vertices of $N(v_i)$ for some v_i, where $v_i \in V(C_4(t))$, where $\deg(v_i)$ = 2 in $C_4(t)$. Therefore, D' contains atleast 3 + 2 (t–1) vertices. Hence $|D'| \geq 2t + 1$ and $\gamma_{ctnd}(C_4(t)) = 2t + 1$.

Case 3: $n \geq 5$

Let $D = \cup_{i=1}^{t}\{v_3^i, v_4^i, \ldots, v_n^i\}$. Then D is a dominating set of $C_n(t)$. Also $< V(C_n(t)) - D > \cong P_t \circ K_1$, $N(v_k^i) \subseteq D$, k = 4, 5, …, n – 1, i = 1, 2, …, t and hence D is a ctnd-set of $C_n(t)$ and hence, $\gamma_{ctnd}(C_n(t)) = |D| \leq (n-2)t$.

$\gamma_{ctnd}(C_n) = n - 2$, $n \geq 5$. But there are t cycles and $< \{v_1^1, v_1^2, \ldots, v_1^t\} > \cong P_t$. Therefore, $\gamma_{ctnd}(C_n(t)) \geq t(n-2) - (t-1) \geq t(n-3) + 1$.

Hence $\gamma_{ctnd}(C_n(t)) = t(n-3) + 1$.

Example 3.2.

For the $C_5(5)$ graph given in Figure 3.2, $D = \cup_{i=1}^{5}\{v_3^i, v_4^i, v_5^i\}$ is a minimum ctnd-set of $C_5(5)$ and hence $\gamma_{ctnd}(C_5(5)) = 15$.

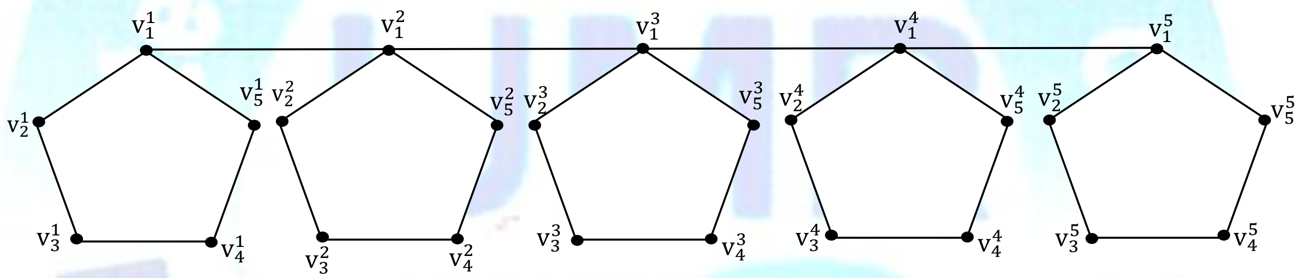

Figure 3.2

Definition 3.3.

A t-ply $P_t(u,v)$ is a graph with t paths joining vertices u and v, each of length atleast two and no two paths have a vertex in common except the end vertices u and v in $P_t(u,v)$.

Theorem 3.3.

$\gamma_{ctnd}(P_t(u, v)) = p - t$, where p is the number of vertices in $P_t(u,v)$.

Proof.

Let the vertices of t path $P^{(i)}$, i = 1, 2, …, t be u, v_1^t, v_2^t, …, v_n^t, v ($n \geq 2$). The vertices u and v are of degree t and the remaining vertices are of degree 2.

Here $|V(P_t(u, v))|$ = nt + 2. Let $D = \cup_{t=1}^{n}\{v_1^t, v_2^t, \ldots, v_{n-1}^t\} \cup \{u, v_n^1\}$. Then D is a dominating set of $P_t(u, v)$ and $<V(P_t(u, v)) - D> \cong K_{1,t-1}$, $N(u) \subseteq D$ and hence, D is a minimum ctnd-set of G and hence, $\gamma_{ctnd}(G) = |D| = (n-1)t + 2 = nt - t + 2 = p - t$.

Let D' be a ctnd-set of $P_t(u, v)$. Then $< V(P_t(u, v)) - D'>$ is a tree. If $<V(P_t(u, v)) - D'>$ contains a path P_4, then D' is not a dominating set of G. Therefore, $<V(P_t(u, v)) - D'>$ is a star and hence $|D'| \geq p - t$. $\gamma_{ctnd}(P_t(u, v)) \geq p - t$.

Hence $\gamma_{ctnd}(P_t(u, v)) = p - t$.

Example 3.3.

For the graph given in Figure 3.3, D = $\bigcup_{t=1}^{6}\{v_1^t, v_2^t\} \cup \{u, v_3^1\}$ is a minimum ctnd-set of G and hence, $\gamma_{ctnd}(G) = |D| = 14$.

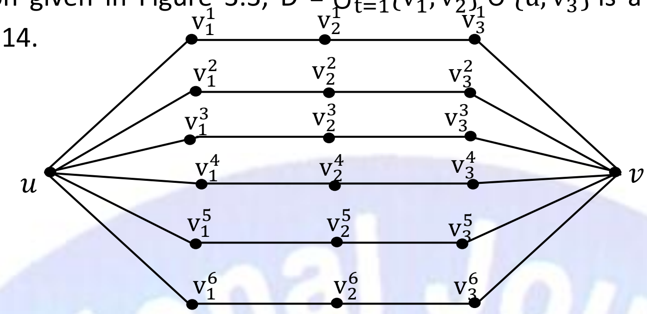

Figure 3. 3

Theorem 3.4.

Let G be a graph such that both G and its complement $\overline{G}$ are connected. Then,

(i) $6 \le \gamma_{ctnd}(G) + \gamma_{ctnd}(\overline{G}) \le 2(p-1)$

(ii) $9 \le \gamma_{ctnd}(G) \cdot \gamma_{ctnd}(\overline{G}) \le (p-1)^2$

Proof.

Let both G and $\overline{G}$ be connected. By Theorem 2.2, $2 \le \gamma_{ctnd}(G)$. But if $\gamma_{ctnd}(G) = 2$ if and only if G is a graph obtained by attaching a pendant edge at a vertex of degree p - 2 in $T + K_1$, where T is a tree on (p – 2) vertices, then $\overline{G}$ has an isolated vertex. Therefore, $3 \le \gamma_{ctnd}(G)$. Hence $6 \le \gamma_{ctnd}(G) + \gamma_{ctnd}(\overline{G})$ and $9 \le \gamma_{ctnd}(G) \cdot \gamma_{ctnd}(\overline{G})$

By Theorem 2.5, $\gamma_{ctnd}(G) = p$ if and only if G is a complete graph on p vertices. But in this case $\overline{G}$ is disconnected. Therefore, $\gamma_{ctnd}(G) \le p-1$. Hence, $\gamma_{ctnd}(G) + \gamma_{ctnd}(\overline{G}) \le 2(p-1)$ and $\gamma_{ctnd}(G) \cdot \gamma_{ctnd}(\overline{G}) \le (p-1)^2$.

Both lower and upper bounds are attained, if G is a path on 4 vertices.

4. Relationship between Complementary Tree Nil Domination Number and Other related Parameters

Theorem 4.1.

For any connected graph G, $d(G) \le \gamma_{ctnd}(G)$, where d(G) is the domatic number of G.

Proof.

By Theorem 2.3, for any connected graph G, $\delta(G) + 1 \le \gamma_{ctnd}(G)$. By Theorem 2.1, for any connected graph G, $d(G) \le \delta(G) + 1$. Hence $d(G) \le \gamma_{ctnd}(G)$.

Equality holds, if $G \cong K_p$.

Theorem 4.2.

For any connected graph G with p vertices ($p \ge 3$), $d_{ctnd}(G) = 1$.

Proof.

By the definition of complementary tree nil dominating set, if D is a ctnd-set, then V – D is not a dominating set. Exactly one domatic partition can be constructed in ctnd- set for any connected graph G. Therefore, any two ctnd-sets of G are not disjoint and hence, $d_{ctnd}(G) = 1$.

Theorem 4.3.

For any connected graph G, $G \ncong K_{1, p-1}$, $\gamma_{ctnd}(S(G)) = p + q - \Delta(G)$, where S(G) is the subdivision graph of G.

Proof.

Let G be not a star. Let v be a vertex of maximum degree in G and let u be a subdivided vertex adjacent to v in S(G).

If D = $N_{s(G)}[v] - \{u\}$, then V(S(G)) – D is a dominating set of S(G), $< D >_{S(G)} \cong K_{1,\Delta(G)-1}$ and $|D| = \Delta(G)$. Since G is a not a star, there exists an edge say e not incident with v in G. Then w be the subdivided vertex of the edge e in S(G). Therefore, w ϵ V(S(G)) – D such that $N_{s(G)}(w) \subseteq V(S(G)) - D$. Therefore, V(S(G)) – D is a ctnd-set of S(G) and hence $\gamma_{ctnd}(S(G)) \leq |V(S(G)) - D|$ = p + q - Δ(G).

Let D′ be a ctnd-set of G. Then < V(S(G)) – D′> is a tree. If < V(S(G)) – D′> contains a path P_4, then D′ is not a dominating set of G. Therefore, < V(S(G)) – D′> is a star on Δ(G)vertices in G and hence |D′| ≥ p + q - Δ(G). $\gamma_{ctnd}(S(G)) \geq p + q - \Delta(G)$.

Hence $\gamma_{ctnd}(S(G)) = p + q - \Delta(G)$.

Theorem 4.4.

Let T be a tree. Then $\gamma_{ctnd}(T) = \gamma_{ctd}(T) + 1$ if and only if each vertex of degree atleast 2 in T is a support.

Proof.

Let S = { $v_1, v_2, \ldots, v_m$} be a pendant vertices in T and |S| = m. Let D and D′ be ctd and ctnd sets respectively.

S is a γ_{ctd}- set of T if and only if every vertex of T is a support. Let $v_i \in$ <V(T) – S > be an end vertex of <V(T) –S>. Then S ∪N(v_i) a γ_{ctnd} – set of T if and only if each vertex of degree atleast 2 is a support.

Remark 4.1.

$\gamma_{ctnd}(T) = \gamma_{ctd}(T)$ if and only if $\gamma_{ctd}(T) > m$, where m is the set of all pendant vertices in T.

Theorem 4.5.

For any connected graph G, $\gamma_{ctnd}(G) + \Delta(G) \leq 2p - 1$.

Proof.

For any graph with p vertices Δ(G) ≤ p − 1, by the Theorem 2.2, $\gamma_{ctnd}(G) \leq p$ which implies $\gamma_{ctnd}(G) + \Delta(G) \leq 2p-1$.

Theorem 4.6.

For any connected graph G, $\gamma_{ctnd}(G) + \Delta(G) = 2p - 1$ if and only if G ≅K_p.

Proof.

If G ≅K_p, then $\gamma_{ctnd}(G) + \Delta(G) = 2p - 1$.

Conversely, assume $\gamma_{ctnd}(G) + \Delta(G) = 2p - 1$. Then the only possible case is $\gamma_{ctnd}(G) = p$ and Δ(G) = p - 1.

But $\gamma_{ctnd}(G) = p$ if and only if G ≅K_p. and Δ(G) = p - 1.

Hence G ≅K_p.

Theorem 4.7.

For any connected graph G, $\gamma_{ctnd}(G) + \Delta(G) = 2p - 2$ if and only if G is one of the following graphs:

$K_{1,p-1}$, G is isomorphic to a graph K_p – Y, where Y is a matching in K_p by joining atleast one edge e ∈ Y.

Proof.

Let $\gamma_{ctnd}(G) + \Delta(G) = 2p - 2$. Then either

(i) $\gamma_{ctnd}(G) = p$ and $\Delta(G) = p - 2$ or

(ii) $\gamma_{ctnd}(G) = p - 1$ and $\Delta(G) = p - 1$.

Case 1. $\gamma_{ctnd}(G) = p$ and $\Delta(G) = p - 2$

By Theorem 2.5, $G \cong K_p$. But for a complete graph, $\Delta(G) = p - 1$ and hence this case is not possible.

Case 2. $\gamma_{ctnd}(G) = p - 1$ and $\Delta(G) = p - 1$.

$\gamma_{ctnd}(G) = p - 1$ if and only if G is one of the following

(i) $K_{1, p-1}$ or $S_{m,n}(m + n = p, m, n \geq 2)$, if $\delta(G) = 1$.

(ii) G is a graph in which each edge is a dominating edge, if $\delta(G) \geq 2$.

Subcase 2.1: Let $G \cong K_{1, p-1}$ or $S_{m,n}$, $(m + n = p, m, n \geq 2)$.

In $S_{m,n}$, $\Delta(G) \neq p - 1$. Hence $G \cong K_{1, p-1}$.

Subcase 2.2: G is a graph in which each edge is a dominating edge.

If $\Delta(G) = p - 1$, then G is isomorphic to graph $K_p - Y$, where Y is a matching in K_p by joining atleat one edge $e \in Y$.

Hence G is isomorphic to $K_{1,p-1}$ or to a graph $K_p - Y$, where Y is a matching in K_p by joining atleast one edge $e \in Y$.

Conversely if G is isomorphic to $K_{1, p-1}$ or to a graph $K_p - Y$, where Y is a matching in K_p by joining atleast one edge $e \in Y$, then $\gamma_{ctnd}(G) = p - 1$ and $\Delta(G) = p - 1$. Hence $\gamma_{ctnd}(G) + \Delta(G) = 2p - 2$.

Theorem 4.8.

For any connected graph G, $\gamma_{ctnd}(G) + \Delta(G) = 2p - 3$, if and only if G is one of the following graphs:

1. $S_{2,n}$, $(n \geq 2)$, $K_{2,n}$, $(n \geq 2)$, W_6.
2. G is a graph obtained from a complete graph by attaching pendent edges at exactly one vertex.
3. G is a graph which is the one point union of complete graphs.
4. G is a graph such that there exists a vertex $v \in V(G)$ such that $G - v$ is a complete graph on $(p - 1)$ vertices and $G \not\cong K_p$.
5. G is a graph such that there exists a vertex $v \in V(G)$ such that $G - v$ is $K_{p-1} - e$, $(e \in E(K_{p-1}))$ and $N(v)$ contains atleast one vertex of degree $(p - 3)$ in $K_{p-1} - e$.

Proof.

If G is a graph stated in the theorem, then $\gamma_{ctnd}(G) + \Delta(G) = 2p - 3$.

Conversely, assume $\gamma_{ctnd}(G) + \Delta(G) = 2p - 3$. Then there are three cases to consider

(i) $\gamma_{ctnd}(G) = p$ and $\Delta(G) = p - 3$ and

(ii) $\gamma_{ctnd}(G) = p - 1$ and $\Delta(G) = p - 2$

(iii) $\gamma_{ctnd}(G) = p - 2$ and $\Delta(G) = p - 1$

Case 1. $\gamma_{ctnd}(G) = p$ and $\Delta(G) = p - 3$

By Theorem 2.5, $G \cong K_p$. But for a complete graph, $\Delta(G) = p - 1$ and hence this case is not possible.

Case 2. $\gamma_{ctnd}(G) = p - 1$ and $\Delta(G) = p - 2$

$\gamma_{ctnd}(G) = p - 1$ if and only if G is one of the following graph:

(i) $K_{1,\,p-1}$ or $S_{m,n}(m + n = p,\ m, n \geq 2)$, if $\delta(G) = 1$.

(ii) G is a graph in which each edge is a dominating edge, if $\delta(G) \geq 2$.

Subcase 2.1. Let $G \cong K_{1,\,p-1}$ or $S_{m,n}$, $(m + n = p,\ m, n \geq 2)$.

If $G \cong K_{1,\,p-1}$, then $\Delta(G) \neq p - 2$.

If $G \cong S_{m,n}$, then $\Delta(G) = p - 2$ is possible only if $G \cong S_{2,n}\ (n \geq 2)$.

Hence $G \cong S_{2,\,n}\ (n \geq 2)$.

Subcase 2.2. G is a graph in which each edge is a dominating edge.

Since each edge of G is a dominating edge, every vertex of X is adjacent to all the vertices in V – X. Let X = {x, y} and let V – X = $\{v_1, v_2, \ldots, v_{p-2}\}$.

Subcase 2.2.1 x and y are adjacent

Then $\Delta(G) = p - 1$. But $\Delta(G) = p - 2$. Therefore no graph exists in this case.

Subcase 2.2.2 x and y are non adjacent.

If each edge of <V – X> is a dominating edge, then $\Delta(G) = p - 1$. But $\Delta(G) = p - 2$.

If there exists atleast one edge in < V - X > which is not a dominating edge, then $\gamma_{ctnd}(G) = p - 2$. But $\gamma_{ctnd}(G) = p - 1$.

If each edge of < V – X > is independent, then $G \cong K_{2,n}$, $(n \geq 2)$.

Case 3. $\gamma_{ctnd}(G) = p - 2$ and $\Delta(G) = p - 1$

By Theorem 2.8, Theorem 2.10, and Theorem 2.11, $\gamma_{ctnd}(G) = p - 2$ if and only if G is one of the following graphs:

1. $G \cong T$, where T is a tree either

obtained from a path P_n ($n \geq 4$ and $n < p$) by attaching pendant edges at atleast one of the end vertices of P_n.

or

obtained from P_3 by attaching pendant edges at either both the end vertices or all the vertices of P_3.

2. $G \notin \mathcal{G}$, if $\delta(G) = 1$

3. If $\delta(G) \geq 2$, then G is one of the following graphs.

(i) A cycle on atleast five vertices.

(ii) A wheel with six vertices.

(iii) G is a graph which is the one point union of complete graphs.

(iv) G is obtained by joining two complete graphs by an edge.

(v) G is a graph such that there exists a vertex $v \in V(G)$ such that $G - v$ is complete graph on $(p - 1)$ vertices and $G \ncong K_p$.

(vi) G is a graph such that there exists a vertex $v \in V(G)$ such that $G - v$ is $K_{p-1} - e$, $(e \in E(K_{p-1}))$ and N(v) contains atleast one vertex of degree $(p - 3)$ in $K_{p-1} - e$.

Case 3.1. $G \cong T$,

In this case $\Delta(G) \neq p - 1$. Since $G \ncong K_{1,p-1}$. Therefore no connected graph exists in this case.

Case 3. 2. $G \notin \mathcal{G}$ and $\delta(G) = 1$

G is a graph obtained from a complete graph by attaching pendent edges at exactly one vertex.

Case 3.3. $\delta(G) \geq 2$.

Subcase 3.3.1.A cycle on atleast five vertices.

$\Delta(C_p) = 2$ and $\Delta(G) = p - 1$ implies $p = 3$. But $p \geq 5$. Therefore no connected graph exists in this case.

Subcase 3.3.2. G is obtained by joining two complete graphs by an edge.

In this case $\Delta(G) \neq p - 1$. Therefore no connected graph exists in this case.

Subcase 3.3.3. A wheel with six vertices, G is the one point union of complete graphs, G is a graph such that there exists a vertex $v \in V(G)$ such that $G - v$ is a complete graph on $(p - 1)$ vertices and $G \ncong K_p$ or G is a graph such that there exists a vertex $v \in V(G)$ such that $G - v$ is $K_{p-1} - e$, $(e \in E(K_{p-1}))$ and N(v) contains atleast one vertex of degree $(p - 3)$ in $K_{p-1} - e$.

In these cases, $\Delta(G) = p - 1$.

Hence G is isomorphic to one of the following graphs

1. $S_{2,n}$, $(n \geq 2)$, $K_{2,n}$, $(n \geq 2)$, W_6.
2. G is a complete graph by attaching pendent edges at exactly one vertex.
3. G is a graph which is the one point union of complete graphs.
4. G is a graph such that there exists a vertex $v \in V(G)$ such that $G - v$ is complete graph on $(p - 1)$ vertices and $G \ncong K_p$.
5. G is a graph such that there exists a vertex $v \in V(G)$ such that $G - v$ is $K_{p-1} - e$, $(e \in E(K_{p-1}))$ and N(v) contains atleast one vertex of degree $(p - 3)$ in $K_{p-1} - e$.

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