Connected Total Global Domination in Fuzzy Graphs

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Abstract
A connected total dominating set D of fuzzy graph G is a connected total global dominating set if D is also a connected total dominating set of $\bar{G}$. The minimum cardinality of a fuzzy connected total global dominating set of G is the fuzzy connected total global domination number γ_{fctg}(G). In this paper we study a connected total global domination in fuzzy graphs and investigate the relationship of with other known parameters.
Keywords: Fuzzy Graph, Fuzzy Connected Total Domination Set, Fuzzy Connected Global Dominating Set, Fuzzy Connected Total Global Domination Number.

1.INTRODUCTION

The study of dominating sets in graphs was begun by Orge and Berge, V.R.Kulli wrote on theory of domination in graphs. The capability of fuzzy sets to express gradual transition from membership to non-membership and vice versa has a broad utility. It provides us not only with a meaningful and powerful representation of measurement of uncertainties, but also with a meaningful representation of vague concepts expressed in natural language. Because every crisp set is fuzzy but not conversely, the mathematical embedding of conventional set theory into fuzzy sets is as natural as the idea of embedding the real numbers into the complex plane. Thus, the idea of fuzziness is one of enrichment, not of replacement. The domination number is introduced by cockayne and HedetniemiResenfeld introduced the notion of fuzzy graph and several fuzzy analogs of graph theoretic concepts such as paths, cycles and connectedness etc. A.Somasundaram and S.Somasundaram discussed domination fuzzy graph NagoorGani and Chandrasekaran discussed domination in fuzzy graph using strong arc. NagoorGani and Vadivel discussed domination, independent domination and irredundant in fuzzy using strong arcs. This concept of total global domination number was introduced by V.R.Kulli and M.B.Kattiman. Among the various applications to the theory of domination in fuzzy graphs, here we consider the fault tolerant property in communication network, that is, even if any communication link to a station is failed, still it can communicate the message to that station. We also discuss the fuzzy connected total global domination number of the fuzzy graph.

2. PRELIMINARIES:

A fuzzy subset of a non-empty set V is a mapping $\sigma : V \rightarrow [0,1]$.A fuzzy relation on V is a fuzzy subset of $V \times V$. A *fuzzy graph* G = (σ , μ) is a pair of function $\sigma : V \rightarrow [0,1]$ and $\mu : V \times V \rightarrow [0,1]$, where $\mu (u ,v) \leq \sigma(u) \Lambda \sigma (v)$ for all $u ,v \in V$. The fuzzy graph $H=(\tau ,\rho)$ is called a fuzzy *subgraph*of G = (σ,μ) if $\tau(u) \leq \sigma(u)$ foe all $u \in V$ and $\rho (u ,v) \leq \mu (u ,v)$ for all $u ,v \in V$. A fuzzy subgraph of $H=(\tau ,\rho)$ is said to be a *Spanning fuzzy subgraph* of G = (σ,μ) if $\tau(u) \leq \sigma(u)$ for all u. Let G = (σ,μ) be a fuzzy graph and τ be any fuzzy subset of σ,(ie) $\tau(u) \leq \sigma(u)$ for all u. Then the fuzzy subgraph of G = (σ,μ) *induced* by τ is the maximal fuzzy subgraph of G = (σ,μ) that has fuzzy nodes set τ. A set $D \subseteq V$ of a fuzzy graph G is said to be independent if $\mu(u,v) \le \sigma(u)\Lambda\sigma(v)$ for all u ,v in D. The *order p* and *size q* of the fuzzy graph G = (σ, μ) are define by p = $\sum_{v\in V}\sigma(v)$and q = $\sum_{u,v\in E}\mu(u,v)$. The *complement* of a fuzzy graph G = (σ , μ) is a fuzzy graph G = (σ , μ) where σ = σ and $\mu^c(u,v) = \sigma(u)\ \Lambda\sigma(v) - \mu(u,v)$ for all u ,v in V. The *strength* of connectedness between two nodes u ,v in a fuzzy graph G is $\mu^{\infty}(u,v) = \sup\{\mu^k(u,v): k=1,2,3\ldots\ldots\}$ where $\mu^k(u,v) = \sup\{\mu(u,u_1)\Lambda\mu(u_1,u_2)\Lambda \ldots\ldots\Lambda\mu(u_{k-1},v)\}$, An arc (u ,v) is said to be a *strong arc* if $\mu(u,v) \ge \mu^{\infty}(u,v)$ and the node V is said to be a strong neighbor of u. If μ(u ,v) = 0 for every v∈V, then u is called *isolated node*. The fuzzy graph H(τ, ρ) is called a *partial fuzzy subgraph* of G(σ,μ) if τ ≤σ and ρ ≤μ. The partial fuzzy subgraph of G(σ,μ) induced by τis maximal partial fuzzy subgraphof G that has fuzzy vertex set τ. This is partial fuzzy graph H(τ, ρ) where $\rho(x,y) = \tau(x) \wedge \tau(y) \wedge \mu(x,y)\ \forall x,y\in V$. An arc (u,v) is called a *fuzzy bridge* in G, if the removal of (u,v) reduces the strength of connectedness between some pair of nodes in G. A connected fuzzy graph is called a fuzzy tree, if it contains a spanning subgraphs T which is a tree such that for all arcs (u,v) not in T. A node cover of a graph G is a nodes that covers all the sizes and an size cover of G is a set of sizes that covers all the nodes. The node (size) covering number $(\alpha_0(G),\alpha_1(G))$ of G is minimum cardinality of a node (size) cover. A set S of nodes of G is independent if no two nodes in S are adjacent. The independence number $\beta_0(G)$ of G is the maximum cardinality of an independent set. A β_0 set is a maximum independent set. A set F of nodes of G is independent if no two sizes in F are adjacent. The size independence number $\beta_1(G)$ of G is the maximum cardinality among the independent sets of sizes. A set $S \subseteq V$ in a fuzzy graph G is said to be*independent* if $\mu(u,v) < \sigma(u)\Lambda\sigma(v)$ for all u ,v ∈ S. A dominating set is called an*independent dominating set* if D is independent. An independent dominating set S of a fuzzy graph G is said to be a*maximal independent dominating set* if there is no independent dominating set S^1 of G such that $S^1 \subset S$. An independent dominating set S of a fuzzy graph G is said to be a*maximum independent dominating set* if there is no independent dominating set S^1 of G such that $|S^1| > |S|$.The minimum scalar cardinality of an maximum independent dominating set of G is called the*independent domination number* of G and is denoted by i(G). Let G be a fuzzy graph and u be a node in G then there exists node v such that (u ,v) is a strong arc then *u dominates v*. A fuzzy graph G = (σ, μ) is said to be *connected* if there exists a strongest path between any two nodes of G. A subset S of V is called a *dominating set* in G if for every v ∉S , there exists u ∈ S such that u dominates v. The minimum fuzzy cardinality of a dominating set in G is called the *domination number* of G and is denoted by γ(G). Let G = (σ, μ) be a fuzzy graph. A subset D of V is said to be *fuzzy dominating set* of G if for every v ∈ V-D, there exists u ∈ D such that (u ,v) is a Strong arc. A fuzzy dominating set D of a fuzzy graph G is called *minimal dominating set* of G, if for every node v ∈ D, D - {v} is not a dominating set. The *fuzzy domination number*$\gamma_f(G)$ is the minimum cardinalities taken over all minimal dominating sets nodes of G. A *connected dominating set* S to be a dominating set S whose induced subgraph<S> is connected. Since a dominating set must contain atleast one node from each component of G. The minimum cardinality of a connected dominating set is the *connected domination number*$\gamma_c(G)$. A *total dominating sets* D of G is a dominating set such that the induced subgraph<D> has no isolates. The *total domination number*$\gamma_t(G)$ of G is the minimum cardinality of a total dominating set. This concept was introduction in cockayneet al[1]. A dominating sets D of G is a *global dominating set* if D is also a dominating set $\bar{G}$ .The *global domination number*$\gamma_f(G)$ of G is the minimum cardinality of a global dominating set (sampathkumer[5]). The purpose of this paper is to study the global aspect of total domination. A total dominating set D of G is a total global dominating set if D is also a total dominating set of $\bar{G}$. The *total global domination number* $\gamma_{tg}(G)$ of G is minimum cardinality of a total global dominating set. Let G: (σ ,μ) be a fuzzy graph. The *degree* of a node u is $d_G(u)= \sum_{u\neq v}\mu(uv)$. Since μ(uv) > 0 for uv∈ E and μ(uv) = 0 for uv∉E, This is equivalent to $d_G(u)= \sum_{uv\in E}\mu(uv)$. The *minimum degree* of G is $\delta(G) = \wedge\{d(v) / v\in V\}$. The *maximum degree* of G is $\Delta(G) = \vee\{d(v) / v\in V\}$.

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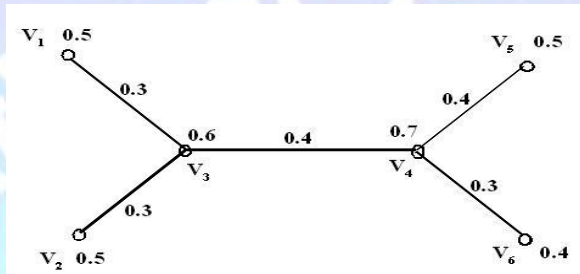
Main Results

3. Fuzzy Connected Total Global Domination number:

Definition: 3.1

A connected total dominating set D of fuzzy graph G is a connected total global dominating set if D is also a connected total dominating set of $\bar{G}$. The minimum cardinality of a fuzzy connected total global dominating set of G is the fuzzy connected total global domination number $\gamma_{fctg}(G)$. In this paper we study a connected total global domination in fuzzy graphs and investigate the relationship of with other known parameters.

Example: 3.2



In the above example $\{v_3, v_4\}$ is a total global dominating set which is also connected in fuzzy graph G. Therefore $\{v_3, v_4\}$ is a fuzzy connected total global dominating set of G.

Theorem: 3.3

A total dominating set D of fuzzy graph G is a fuzzy connected total global dominating set if and only if for each node $v \in V$ there exists a node $u \in D$ such that v is not adjacent to u.

Observation: 3.4

Let G be a fuzzy graph of order $p \geq 2$.

(i) If G has no isolated nodes, then $\gamma_{fctg}(G) \leq p - \Delta(G) + 1$.

(ii) If G is connected fuzzy graph and $\Delta(G) \leq p-2$, then $\gamma_{fctg}(G) \leq p - \Delta(G)$.

Proposition: 3.5

For any fuzzy graph G, $\gamma_{fctg}(G) \leq 2ir(G)$.

Proposition: 3.6

If each component of fuzzy graph G has at least three nodes, then $i(G) + \gamma_{fctg}(G) \leq p$.

Theorem: 3.7

If H is a connected spanning fuzzy subgraph of G, then $\gamma_{fctg}(G) \leq \gamma_{fctg}(H)$.

Note: Let $\in_T$ be the maximum number of pendant nodes any spanning tree. He defniemi and Laskar gave an analogus result for the connected domination number.

Theorem: 3.8

If G is a connected fuzzy graph and $p \geq 3$, Then $\gamma_{fctg}(G) = p - \in_T(G) \leq p-2$.

Proof:

Let T be a spanning tree of G with $\in_T(G)$end nodes, and let L denote the set of end nodes. Then T–L is a connected fuzzy dominating set having $p - \in_T$ nodes, that is $\gamma_c(G) \leq p - \in_T(G)$. Conversely , let S be a γ_{fctg} - set. Since <S> is connected, <S> has a spanning tree Ts. A Spanning tree T of G is formed by adding the remaining $p - \gamma_{fctg}(G)$ vertices of V–S to Ts and adding sizes of G, Such that each node in V–S is adjacent to exactly one node in S. Now T has at least $p - \gamma_{fctg}(G)$ end nodes. Thus, $\in_T(G) \geq p - \in_T(G)$ $or \gamma_{fctg} G) \geq p - \in_T(G)$.

Hence, $\gamma_{fctg}(G) = p - \in_T(G)$and since $\in_T(G) \geq 2$.

Theorem: 3.9

A connected total dominating set D of a connected fuzzy graph G is a fuzzy connected total global dominating set if and only if the following conditions are satisfied

(i) Every node in V–D is not adjacent to some node in D.
(ii) There exists a set S $\subseteq$ D such that diam(<S>) $\geq$ 3 and every node in D is not adjacent to some node in S.

Proposition: 3.10

For any connected fuzzy graph G of order p $\geq$ 6 then, γ_{fctg}(G) $\leq$ p–2.

Proposition: 3.11

Let G be a fuzzy graph such that neither G nor $\overline{G}$ have an isolated node. Then 2q– p (p–3) $\leq \gamma_{fctg}$ (G).

Theorem: 3.12

For any fuzzy graph G without isolated nodes, (2q–p (p–3))/2$\leq \gamma_{fctg}$(G) $\leq$p–β_o(G)+1.

Proof:

Let D be a minimum global dominating set. Then every node in V-D is not adjacent to at least one node in D. This implies q $\leq \frac{p(p-1)}{2}$ (p $- \gamma_{fctg}$) and the lower bound follows.

To establish upper bound, let S be an independent set with β_o nodes. Since G has no isolated nodes, V–S is a dominating set of G.

Clearly, for any vertex v $\in$S, (V-S) $\cup${v} is a global dominating set of G and the upper bound follows.

Proposition: 3.13

For any fuzzy graph G without isolated nodes,

(i) γ(G)+ γ_{fctg}(G) $\leq$p+1,
(ii) i(G)+ γ_{fctg} (G) $\leq$p+1.

Theorem: 3.14

If G is a fuzzy graph such that neither G nor $\overline{G}$ have an isolated node,
Then γ_{fctg} (G) $\leq$ 2 α_o (G).

Proof:

Let S be a node cover of G with |S| = α_o(G). Let S = {u_1,u_2,..........u_s}, S ={u_1} is impossible since u_1 has a non-neighbour$x$, and x has a neighbour which can only be u_1, so s $\geq$ 2. Each u $\in$ S has a neighbour, a non-neighbour, or both, in S–u. If u has no neighbour in S, then choose v $\in$ V-S a neighbour of u. If u has no non-neighbour in S, then choose in V– S a non – neighbour v of u. Thus construct D = {v_1,v_2,.......,$v_{s'}$}, $s' \leq$ s, not all nodes v_i ,$v_j$need be distinct, so |D| $\leq$ |S|. If D = {v_1} and if v_1 is adjacent to each node of S, then add a non-neighbour v_1 to D so that |D| $\geq$ 2. A set S$\cup$D is a fuzzy connected total global dominating set, because x $\in$ V–{S$\cup$D} has at least one neighbour which necessarily belongs to S and each node of D is a non-neighbour of x. A node x $\in$ S has by construction both a neighbour of and a non-neighbour in S $\cup$ D. A node x $\in$ D has a neighbour in S and as |D| $\geq$2,x has a non-neighbour in D. Thus γ_{fctg}(G) $\leq$ |S $\cup$ D| = |S| + |D| $\leq$ 2 |S| = 2α_O(G).

Theorem: 3.15

Let G be a fuzzy graph with diam(G) $\geq$ 5. Then D $\subseteq$ V is a connected total dominating set of G if and only if D is a fuzzy connected total global dominating set.

Proof:

Suppose D is a total dominating set of G. Let u, $v$$\in$$V$ such that d (u, v) $\geq$5. Then D $\cap$N(u) $\neq\phi$ and D $\cap$ N(v) $\neq\phi$. Let $u_1$$\in$ D $\cap$ N (u) and $v_1$$\in$ D $\cap$N(v). Then u_1 and v_1 are not adjacent and further every node in V–{u_1, v_1} is adjacent to at most one of u_1 and v_1. This implies that {u_1, v_1} is a total dominating set of $\overline{G}$ and hence D is a total global dominating set. Converse is obvious.

Theorem: 3.16

For any fuzzy graph with p nodes γ_{fctg}(G)=|S| if and only if G = Kp or $\overline{Kp}$ where |S| is the scalar cardinality of all the nodes in G.

Proof:

Assume G = Kp or $\overline{Kp}$. Claim: γ_{fctg}(G)=|S|. W.k.t. $\overline{Kp}$ has all its nodes as fuzzy dominating set and this is the only dominating set of $\overline{Kp}$ Let $\{v_1,v_2..v_i,v_{i+1},.......v_p\}$be the nodes of G. Suppose $\{v_i,v_{i+1},...v_p\}$ is the dominating set of G but it is not the dominating set of $\overline{G}$. So the only dominating set of both G and $\overline{G}$ is the set containing all the nodes of G. So the global dominating set of Kp or $\overline{Kp}$ contains p nodes. Hence γ_{fctg}(G)=|S| . Conversely suppose γ_{fctg}(G)=|S|. To Prove: G= Kpor $\overline{Kp}$.. Suppose G ≠ Kp or $\overline{Kp}$ then G has at least one edge uv and a node w not adjacent to v(say)→V-{v} is global dominating set. So S contains p-1 nodes→γ_{fctg}(G)≠ |S| which is contradiction Hence G ≠ Kp or $\overline{Kp}$.

Theorem 3.17 :

Let G(σ,μ) be a connected fuzzy graph order n and let γ_{fctg}(G) = d ,let S ⊆V be such that <S> is connected then | N(S) –S| ≤ n – d.

Proof:

The result is clear if d ≤ |S| so assume d>|S|. If V–S–N(S) = Φthen γ_{fctg}(G) < d a contradiction. Let H= < V–S–N(S) > and let $H_1,H_2,..H_l$be the connected components of H. Because G is connected for each i,*1* ≤ i ≤*l* there exists u_i∈N (S) and v_i∈V (H_i) Such that u_i is adjacent to v_i where $\mu(u_i, v_i) > 0\ \forall i$. The subgraph induced by X is connected and that X dominates G. Hence d ≤| x | ≤ |v| − |N(S) –S| that is |N(S) –S| ≤ |v| – d.

4. More on the Parameter γ_{fctg}

We present lower and upper bounds on γ_{fctg}(G) for a general graph in terms of its order, and we characterize the graphs attaining these bounds. Fix p ≥ 2 an integer. We define a family *F* of graphs of order n as follows. Fix A and B two disjoint subsets of nodes such that |A ∪ B| = p −2 (A = B = ∅ when p=2), and let a, b∉ A ∪ B. Let V= {a, b}∪ A ∪ B. Denote by F(A, B) the set of graphs with node set V satisfy the following properties.

(1) The node a is joined to b.
(2) The node a is joined to each node in A whenever A ≠∅.
(3) The node b is joined to each node in B whenever B ≠∅.
(4) None of V(G) − {v} is joined to both a and b.

Theorem: 4.1

Let G be a fuzzy graph of order p ≥ 2. Then

(i) 2 ≤γ_{fctg}(G) ≤ p.
(ii) γ_{fctg}(G) = p if and only if G ≅ Kn.
(iii) γ_{fctg}(G) =2 if and only if G∈ Ƒ.

Proof:

For item (i), note that 1 ≤γ_{fctg}(G) ≤p, it suffices to show that γ_{fctg}(G) ≠ 1.To the contrary, we may assume that there exists a fuzzy connected total global dominating set S = {*v*}. Hence, *v* is joined to all nodes in V(G) − {v},and so v is isolated in $\bar{G}$. This contradicts that {v} is a fuzzy connected total global dominating set of G.

For item (ii), note that the graph $\overline{Kp}$consists of n isolated nodes, so the proof of sufficiency is trivial. To show necessity, we take a spanning tree of G, say T. Let v be a leaf of T, and let S= V(G) − {v}. Note that v is joined to some node in S, and the subgraph in G induced by S is connected. Hence, S is a connected dominating set of G with size p −1. As γ_{fctg}(G) = p, S is not a fuzzy connected total global dominating set of G. Therefore, v is isolated in $\bar{G}$implying that v is joined to all nodes of S in G. Now we take another spanning tree T′ consisting of all sizes incident with v. Note that every node u ∈ V(G) − {v}is a leaf of T′, and S′ = V(G) − {u} is a connected dominating set of G with size p −1. By a similar argument, one can show that u is joined to all nodes in S′. Hence, G ≅ Kp.

For item (iii), we prove sufficiency first. Let G = F(A, B) $\in$ Ƒ. By the construction, we know that {a, b} is a fuzzy connected total global dominating set of G. Hence $\gamma_{fctg}(G) \leq 2$. By item (i), $\gamma_{fctg}(G) = 2$. Now we prove necessity. Let G be an arbitrary graph with $\gamma_{fctg}(G) = 2$. Suppose that S={a, b} is a fuzzy connected total global dominating set of G. Then ab$\in$E(G) . Since S is a fuzzy connected total global dominating set of G, there exists no node joined to both a and b. As S is a dominating set of G each node u$\in$V(G) $-$ S must be joined to either a or b. Hence, for each node u $\in$V(G) $-$ S, u is joined to either a or b but not both. Denote by A and B the sets of nodes which are joined to a and b, respectively. Clearly, G = F(A, B), and so G $\in$ Ƒ.

Theorem: 4.2

Let T be a tree of order $p \geq 3$. If T $\ncong K_{1,k}$ where $k \geq 2$ is an integer, then $\gamma_{fctg}(T) = p-\mu(T)$, where μ (T) is the number of leaves of T.

Corollary: 4.3

Let T be a tree of order p $\geq$ 4. There $\gamma_{fctg}(T) = p - 2$ if and only if $T \cong P_p$.

Proof:

We only prove necessity, as sufficiency follows from theorem 4.2. Suppose that T is a tree of order p $\geq$ 4 with $\gamma_{fctg}(T) = p-2$. By theorem 4.2, we know that there are exactly 2 leaves in the tree T. It is straightforward to show that any tree with exactly 2 leaves must be isomorphic to a path on p nodes.

Theorem: 4.4

Let G be a fuzzy graph and let S be a connected domination set of G. Then S contains all cut nodes (if exist) of G.

Proof:

Suppose that there exists a cut node v $\in$V(G) such that v $\notin$ S. Note that S $\subset$ V(G) $-$ {v} and G$-$v is disconnected. This is a contradiction, as S forms a connected dominating set of G.

Proof of Theorem 4.2. Note that T has at least two leaves. Let U be the set of leaves in T. Since T $\ncong K_{1,k}$, T$-$U is nontrivial connected. Note that the subgraph induced by V(T) $-$ U is a tree, so it is connected. As every leaf of T is joined to some node in V(T) $-$ U but not all, V(T) $-$ U is a fuzzy connected total global dominating set of T. Thus, $\gamma_{fctg}(T) \leq p-\mu(T)$. Note that every node in V(T) $-$ U is a cut node in T. By previous theorem, $\gamma_{fctg}(T) \geq p-\mu(T)$.

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