Z_n -MULTIPLICATIVE GRAPHS

Dr Ajitha V

P.G Department of Mathematics,

Mahatma Gandhi College, Iritty-670703, Kerala, India

Silja C

Research Scholar,

Mary Matha Arts & science College, Mananthavady

Kannur University, Kannur- 670 567,Kerala, India

Abstract

Let $G=(V,E)$ be a graph with p vertices and q edges. A graph G is said to admit Z_{q+1} multiplicative labeling if its vertices can be labeled by non negative integers such that induced edge labels obtained by the products of the labels of end vertices modulo $q+1$ are the positive numbers up to *q*. A graph admit such a labeling called Z_{q+1} multiplicative graphs. In this paper we give certain graphs which admit Z_{q+1} multiplicative labeling and certain graphs not admit Z_{q+1} multiplicative labeling.

Keywords

Z_{q+1}-multiplicative labeling, Z_{q+1}-multiplicative graph.

1. Introduction

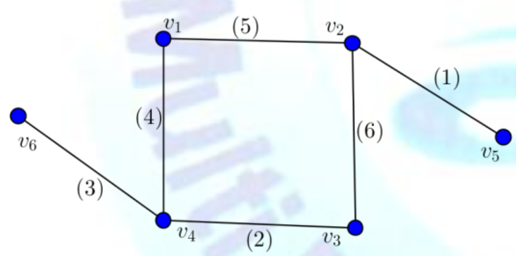
By a graph we mean a finite, connected, simple undirected graph with. The vertex set and the edge set of the graph *G* are denoted by *V*(*G*) and *E*(*G*) respectively. For various graph theoretic notations and terminology we follow F. Harrary [1]

Definition 1.1 Let $G=(V,E)$ be a (p,q) graph. The graph G is said to be a Z_{q+1}-multiplicative labeling if there exist a one-to-one function $f:V(G)\to N$ (where *N* is the set of all positive integers) that induces a function $f^*:E(G)\to Z_{q+1}-\{0\}$ of the edges of G defined by $f^*(uv)=f(u)f(v)\bmod(q+1)$ for every $e=uv\in E(G)$. The graph which admits such a labeling is called a Z_{q+1}-multiplicative graph.

Example 1.2 Z_7-multiplicative graph.

Define $f:V(G)\to N$ by

$f(v_1)=4,\ f(v_2)=3,\ f(v_3)=2,\ f(v_4)=1,\ f(v_5)=5,\ f(v_6)=10.$



Now we discus about star graph.

Theorem 1.3 : Let The star graph K_{1n} is Z_{n+1}-multiplicative graphs.

Proof : Let v_0 be the apex vertex and $v_1,v_2,\ldots,v_n$ be the pendent vertices of the star graph K_{1n}

define $f(v_0)=n+2$

$f(v_i)=i$ for $i=1,2,\ldots,n$.

$E(K_{1n})=\{v_0v_1,v_0v_2,\ldots,v_0v_n\}$.

For $i = 1, 2, ..., n$

$$\begin{aligned} f(v_0)\,f(v_i) &= (n+2)i \bmod (n+1) \\ &= i \bmod (n+1) \\ &= f^*(v_0 v_i) \end{aligned}$$

We see that the induced edge labels obtained by the product of the labels of the vertices are the first n positive integers.

Theorem 1.4 : The cycle C_3 is not Z_4-multiplicative graph.

Proof : Let v_1, v_2, v_3 be vertices of C_3. Define $f : V(G) \to N$ by

$f(v_1) = a,\ f(v_2) = b,\ f(v_3) = c,$ where a, b, c are distinct integers.

Let $f^*(v_1v_2) \equiv 2(\bmod 4)$

that is $f(v_1)\,f(v_2) = ab \equiv 2(\bmod 4)$.

$\Rightarrow a$ or b must be even. $\Rightarrow ac$ or bc must be even.

$\Rightarrow ac = f(v_1)\,f(v_3) = f^*(v_1v_3)$ or $bc = f(v_2)\,f(v_3) = f^*(v_2v_3)$ must be even.

Then either $f^*(v_1v_3)$ nor $f^*(v_2v_3)$ congruent to 1 or 3 modulo 4.

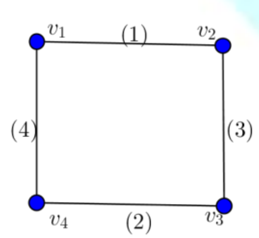
Which contradict definition of f^*.

$\Rightarrow C_3$ is not Z_4-multiplicative graph.

Theorem 1.5 : The cycle C_4 is Z_5-multiplicative graph.

Proof : Define $f : V(G) \to N$, by

$f(v_1) = 1,\ f(v_2) = 6,\ f(v_3) = 3,\ f(v_4) = 4$



Theorem 1.6 : The cycle C_5 is not a Z_6-multiplicative graph.

Proof : Let v_1, v_2, v_3, v_4, v_5 be vertices of C_5. Define $f : V(G) \to N$ by

$f(v_1) = a,\ f(v_2) = b,\ f(v_3) = c,,\ f(v_4) = d,\ f(v_5) = e,$, where a, b, c, d, e are distinct integers.

Let $f^*(v_1v_2) \equiv 3(\bmod 6)$

that is $f(v_1)f(v_2) = ab \equiv 3(\bmod 6)$

$\Rightarrow$ a or b must be a multiple of 3. Let it be a.

Now,

if a is even $\Rightarrow$ $a \equiv 0(\bmod 6) \Rightarrow ab \equiv 0 \bmod 6$

if b is even $\Rightarrow$ $ab \equiv 0 \bmod 6)$

that is $f^*(v_1v_2) = f(v_1)f(v_2) = ab \equiv 0(\bmod 6)$. Which is not possible.

Theorem 1.7 : Complete graph K_3 is not Z_4- multiplicative .

Proof : We know that $K_3 = C_3$.

Already proved that C_3 is not Z_4- multiplicative graph.

Theorem 1.8 : For a Complete graph K_n with $(p-1)$ edges where p is a prime number then K_n is not $Z_{p-1}-$ multiplicative graph .

Proof : Let $v_1, v_2 ..., v_n$ be vertices of K_n .

Consider v_i, with $f^*(v_iv_{i+1}) \equiv 1 \bmod (p-1)$

$f(v_i)f(v_{i+1}) = f^*(v_iv_{i+1}) \equiv 1 \bmod (p-1)$

This imply $f(v_i) \equiv 1 \bmod (p-1)$ and $f(v_{i+1}) \equiv 1 \bmod (p-1)$ or

$f(v_i) \equiv -1 \bmod (p-1)$ & $f(v_{i+1}) \equiv -1 \bmod (p-1)$

If $f(v_i) \equiv 1 \bmod (p-1)$ & $f(v_{i+1}) \equiv 1 \bmod (p-1)$

Consider v_j , $j \neq i, i+1$

$f^*(v_iv_j) = f(v_i)f(v_j) \equiv f(v_j) \bmod (p-1)$

$f^*(v_{i+1}v_j) = f(v_{i+1})f(v_j) \equiv f(v_j) \bmod (p-1)$.

Which is not possible

If $f(v_i) \equiv -1 \bmod (p-1)$ & $f(v_{i+1}) \equiv -1 \bmod (p-1)$

Consider v_j , $j \neq i, i+1$

$f^*(v_iv_j) = f(v_i)f(v_j) \equiv -f(v_j) \bmod (p-1)$

$f^*(v_{i+1}v_j) = f(v_{i+1})f(v_j) \equiv -f(v_j)\,\text{mod}(p-1)$

Which is not possible. There for no such labeling exist.

Theorem 1.9 : For a Complete graph K_n with p^2-1 edges where p is a prime number then K_n is not Z_{p^2} – multiplicative graph .

Proof : Let $v_1, v_2 ..., v_n$ be vertices of K_n . Define $f : V(G) \to N$ by $f(v_i) = a_i$ where a_i are distinct integers , for every i .

Since K_n is complete then the edge set $E(G) = \{v_i v_j \;\; ; \;\; 1 \le i, j \le n,\ i \ne j\}$.

Assume K_n is Z_{p^2} – multiplicative graph .

Let $f^*(v_k v_j) = p$

$\Rightarrow f(v_k)f(v_j) \equiv p(\text{mod}\, p^2)$

$\Rightarrow a_k a_j \equiv p(\text{mod}\, p^2)$

Then either a_k or a_j is a multiple of p .

Let it be a_k .

$\Rightarrow a_i a_k$ Is a multiple of p , for every $1 \le i \le n$.

$\Rightarrow f^*(v_i v_k) = a_i a_k$ is a multiple of p , for every $1 \le i \le n$.

$\Rightarrow f^*(v_i v_k)$ has the possibilities $p, 2p, ..., (p-1)p$

there are $p-1$ possibilities . But there are $n-1$ edges ,

Then $p-1 \ge n-1$ must satisfy $\Rightarrow n \le p$

$n(n-1) \le p(p-1)$ (1)

Since K_n has $\dfrac{n(n-1)}{2}$ edges.

$\Rightarrow \dfrac{n(n-1)}{2} = p^2 - 1 \qquad \Rightarrow n(n-1) = 2p^2 - 2$

(1) Imply $2p^2 - 2 = n(n-1) \le p(p-1)$

$p^2 \le 2 - p \le 2 - 2 = 0 \qquad [\because \;\; p \ge 2]$

$p^2 \le 0$ Is a contradiction .

There for K_n is not Z_{p^2} – multiplicative graph .

Theorem 1.10 : For a Complete graph K_n with $p^m - 1$ edges where p is a prime number and $m > 1$ is any integer then K_n is not $Z_{p^m} -$ multiplicative graph .

Proof : For $m = 2$ the result is above theorem.

Now assume $m > 2$.

Let $v_1, v_2 ..., v_n$ be vertices of K_n . Define $f : V(G) \to N$ by $f(v_i) = a_i$ where a_i are distinct integers , for every i .

Since K_n is complete then the edge set $E(G) = \{v_i v_j \;\; ; \;\; 1 \le i, j \le n, \; i \ne j\}$.

Assume K_n is Z_{p^m} – multiplicative graph .

Let $f^*(v_k v_j) = p$

$\Rightarrow f(v_k) f(v_j) \equiv p (\bmod p^m)$

$\Rightarrow a_k a_j \equiv p (\bmod p^m)$

Then either a_k or a_j is a multiple of p .

Let it be a_k .

$\Rightarrow a_i a_k$ Is a multiple of p , for every $1 \le i \le n$.

$\Rightarrow f^*(v_i v_k) = a_i a_k$ is a multiple of p , for every $1 \le i \le n$.

$\Rightarrow f^*(v_i v_k)$ has the possibilities $p, 2p, ..., (p-1)p, p^2, 2p^2, ..., (p-1)p^2, ..., p^{m-1}, ..., (p-1)p^{m-1}$

there are $(n-1)(p-1)$ possibilities . But there are $n-1$ edges ,

Then $(m-1)(p-1) \ge n-1$ must satisfy .

$\Rightarrow n(n-1) \le [(m-1)(p-1)][(m-1)(p-1)+1]$

imply

$$n(n-1) \le p^2(m^2 - 2m + 1) + p(-2m^2 + 5m - 3) + m^2 - 3m + 2 \qquad (2)$$

Since K_n has $\dfrac{n(n-1)}{2}$ edges.

$$\Rightarrow \frac{n(n-1)}{2} = p^m - 1 \qquad \Rightarrow n(n-1) = 2p^m - 2$$

(2) Imply $2p^m - 2 = n(n-1) \le p^2(m^2 - 2m + 1) + p(-2m^2 + 5m - 3) + m^2 - 3m + 2$

$$\Rightarrow \quad 2p^m - p^2(m^2-2m+1) + p(2m^2-5m+3) - (m^2-3m+4) \le 0 \qquad (3)$$

Consider,

$$2p^m - p^2(m^2-2m+1) + p(2m^2-5m+3) - (m^2-3m+4)$$

$$\ge \quad 2^{m+1} - 4(m^2-2m+1) + 2(2m^2-5m+3) - (m^2-3m+4) \qquad [\because p \ge 2]$$

$$\ge \quad 2^{m+1} - m^2 + m + 6 = 2^{m+1} - (m+1)^2 + 3m + 7$$

$$\ge \quad 2^{m+1} - (m+1)^2 .$$

Since $m > 2 \quad \Rightarrow m+1 > 3 \quad \Rightarrow 2^{m+1} > (m+1)^2$

$$\Rightarrow 2^{m+1} - (m+1)^2 > 0$$

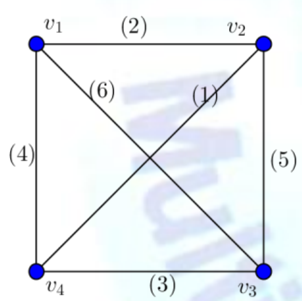
Which imply $\Rightarrow \quad 2p^m - p^2(m^2-2m+1) + p(2m^2-5m+3) - (m^2-3m+4) > 0$

Contradiction to equation (3) .

There for K_n is not Z_{p^m} – multiplicative graph .

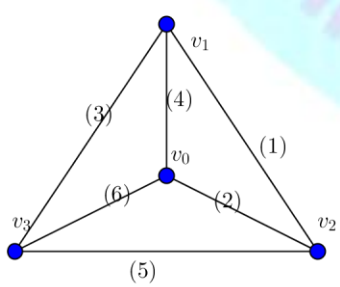
Example 1.11: Complete graph K_4 are Z_7 – multiplicative.

Define $f : V(G) \to N$ by $f(v_1) = 1, f(v_2) = 2, f(v_3) = 6, f(v_4) = 4$



Example 1.12: Wheel W_3 are Z_7 – multiplicative.

Define $f : V(G) \to N$ by $f(v_0) = 1, f(v_1) = 4, f(v_2) = 2, f(v_3) = 6$



Example 1.13: Wheel W_4 is not Z_9 – multiplicative.

Let v_0 be the central vertex and v_1, v_2, v_3, v_4 are vertices of cycle in W_4 .

Define $f: V(G) \rightarrow N$ by $f(v_i) = a_i$, where a_i are distinct integers , for every $0 \le i \le 4$.

Let $f^*(v_1 v_2) \equiv 3(\text{mod}\, 9)$

$\Rightarrow f(v_1) f(v_2) \equiv 3(\text{mod}\, 9)$

$\Rightarrow a_1 a_2 \equiv 3(\text{mod}\, 9)$

Then a_1 or a_2 is a multiple of 3.

Let it be a_1.

$\Rightarrow a_i a_1$ Is a multiple of 3.

v_2, v_4 are adjacent to v_1

$\Rightarrow a_2 a_1$ and $a_4 a_1$ is a multiple of 3.

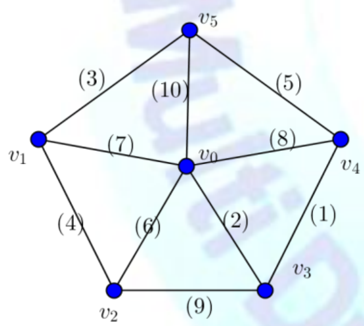
$f^*(v_1 v_2)$ & $f^*(v_1 v_4)$ are multiple of 3

$\Rightarrow f^*(v_1 v_2) \equiv 6(\text{mod}\, 9)$ & $f^*(v_1 v_4) \equiv 6(\text{mod}\, 9)$

Which is a contradiction . So W_4 is not Z_9 – multiplicative.

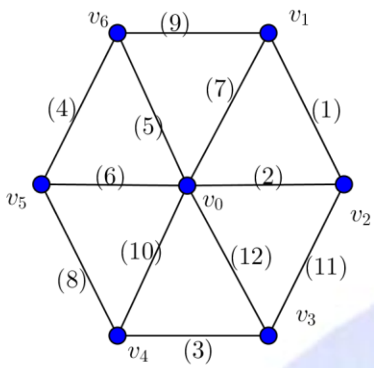
Example 1.14: Wheel W_5 are Z_{11} – multiplicative.

Define $f: V(G) \rightarrow N$ by $f(v_0) = 4, f(v_1) = 10, f(v_2) = 7, f(v_3) = 6, f(v_4) = 2, f(v_5) = 8.$



Example 1.15: Wheel W_6 are Z_{13} – multiplicative.

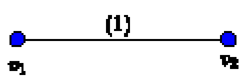
Define $f: V(G) \rightarrow N$ by $f(v_0) = 1, f(v_1) = 7, f(v_2) = 2, f(v_3) = 12, f(v_4) = 10, f(v_5) = 6, f(v_6) = 5.$



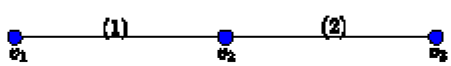
Next consider about paths,

Example 1.16: The path P_n is Z_n – multiplicative graph for $n \leq 10$.

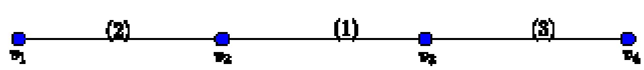
For P_2, $f(v_1)=1, f(v_2)=3$



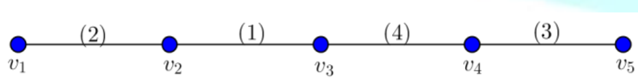
For P_3, $f(v_1)=1, f(v_2)=4, f(v_3)=2$



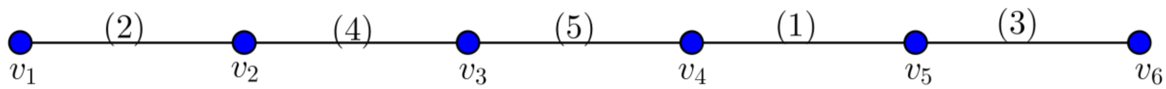
For P_4, $f(v_1)=2, f(v_2)=1, f(v_3)=5, f(v_4)=3$



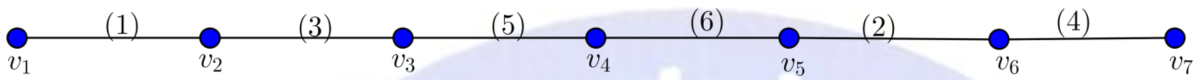
For P_5, $f(v_1)=2, f(v_2)=1, f(v_3)=6, f(v_4)=4, f(v_5)=7$



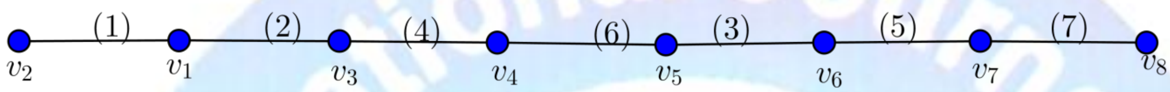
For P_6, $f(v_1)=4, f(v_2)=2, f(v_3)=5, f(v_4)=1, f(v_5)=7, f(v_6)=3$



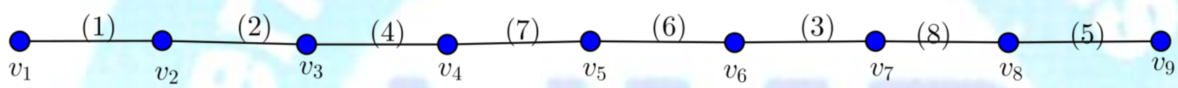
For P_7, $f(v_1)=1, f(v_2)=8, f(v_3)=3, f(v_4)=4, f(v_5)=5, f(v_6)=6, f(v_7)=10$



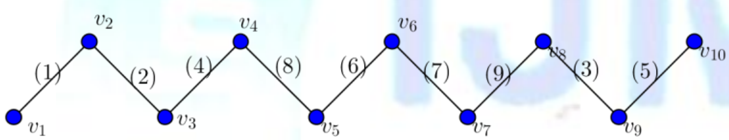
For P_8, $f(v_1)=1, f(v_2)=9, f(v_3)=2, f(v_4)=6, f(v_5)=5, f(v_6)=7, f(v_7)=3, f(v_8)=13$



For P_9, $f(v_1)=1, f(v_2)=10, f(v_3)=2, f(v_4)=11, f(v_5)=8, f(v_6)=3, f(v_7)=7, f(v_8)=5, f(v_9)=19$



For P_{10}, $f(v_1)=1, f(v_2)=11, f(v_3)=2, f(v_4)=7, f(v_5)=4, f(v_6)=9, f(v_7)=3, f(v_8)=13, f(v_9)=21, f(v_{10})=5$



Problem 1.17 The path P_n is Z_n – multiplicative graph for all *n*.

REFERENCES

[1] F. Harary Graph Theory, Addition-Wesley, Reading, Mass, 1972

[2] David M. Burton, Elementary Number Theory, Second Edition, Wm. C. Brown Company Publishers, 1980.

[3] J.A. Gallian, A Dynamic Survey of Graph Labeling, Electronic Journal of Combinatorics, 17 (2010), DS6.