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A Study on Cayley Graphs of Factorizable Inverse Semigroups

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Abstract:
Let $(S,.)$ be a finite semigroup and let T be a non-empty subset of S .The graph $Cay(S,T)$ is defined as the graph whose vertex set is S and edges are the pairs (x,y) such that $tx=y$ for some $t\in T$ and $x\neq y$. Such a graph is called the Cayley graph of S relative to T. In this paper, we characterize and describe some properties of Cayley graphs of Factorizable inverse Semigroup relative to Green's equivalence classes.

Key Words: Cayley graph, complete graph, Factorizable inverse semigroup, Green's Equivalence classes (*L*-class, *R*- class, *H*-class).
AMS Subject Classification: 05C25, 20M18.

1. Introduction

The Cayley graph of groups was introduced by Arthur Cayley in 1878 and the Cayley graphs of groups have received serious attention since then. The Cayley graphs of semigroups are generalizations of Cayley graphs of groups. The whole section 2.4 of the book [7] is devoted to Cayley graphs of semigroups. In 1964, Bosak [2] and in 1981, Zelinka [9] studied certain graphs over semigroups .Recently in 2006, Kelarev [6] studied on Cayley graphs of inverse semigroups. The concept of Factorizable inverse semigroup has been given Chen and Hsieh in [3].The Green's relations play an important role in the theory of semigroups. In this paper, we study the Cayley graphs of Factorizable inverse semigroups relative to Green's equivalence Classes.

2. Preliminaries

In this section we describe some basic definitions and results in Semigroup theory and Graph theory which are needed in the sequel.

Definition 2.1 A pair $(S,.)$ consisting of a non-empty set S and an associative binary operation . on S is called a semigroup.

Definition 2.2 An element x in S is said to be an idempotent if $x^2=x$ and the set of all idempotents in S is denoted as $E(S)$, simply E.

Definition 2.3 (c.f. [5]) Let S be a semigroup. We define $a\,Lb\ (a,b \in S)$ if and only if a and b generates the same principal left ideal, that is, if and only if $S^1a = S^1b$. Similarly we define $a\,Rb$ if and only if a and b generates the same principal right ideal, that is, if and only if $aS^1 = bS^1$. We define $a\,H\,b$ if and only if $a\,Lb$ and $a\,Rb$.

Lemma 2.4 (c. f. [5]) Let a,b be elements of a semigroup S. Then $a\,Lb$ if and only if there exist $x,y \in S^1$ such that $xa = b$, $yb = a$ and $a\,Rb$ if and only if $u,v \in S^1$ such that $au = b,\ bv = a$.

Notation 2.5 The *L*-class (*R*-class, *H*-class) containing an element a in a semigroup S will be written as $L_a\ (R_a, H_a)$.

Definition 2.6 A semigroup S is called an inverse semigroup if every a in S possesses a unique inverse, that is, there exist a unique element a^{-1} in S such that $aa^{-1}a = a$, $a^{-1}aa^{-1} = a^{-1}$.

Theorem 2.7 ([3]) Let S be a semigroup, G be a subgroup of S and $E = E(S)$.Then the following conditions are equivalent

(i) $S = GE$

(ii) $L_e = Ge$ for every $e \in E$.

(iii) $R_e = eG$ for every $e \in E$.

Definition 2.8 A subsemigroup H of a semigroup S is said to be factorizable if there exists a subgroup G and a subset E of idempotents of H such that $H = GE = \{ge / g \in G, e \in E\}$.

Lemma 2.9 ([3]) If the semigroup S is factorizable as $S = GE$,then

(i) $S = EG$

(ii) S has an identity 1 which is the identity of G

(iii) $G = H_1$, the H -class containing 1

(iv) For every $x \in S$ there is a unique $e \in E$ such that $x = ge$ for some $g \in G$ and $E = E(S)$, the set of all idempotents of S.

Definition 2.10 An element x in a semigroup S is called Nilpotent if $x^n = 0$ for some n.

Definition 2.11 A graph G^* is a pair (V, E) where V is a non-empty set whose elements are called vertices of G^* and E is a subset of $V \times V$ whose elements are called edges of G^*. The vertex set of a graph G^* is denoted by $V(G^*)$ and edge set is denoted by $E(G^*)$.

Definition 2.12 A subgraph $H^* = (U, F)$ of a graph $G^* = (V, E)$ is said to be vertex induced subgraph if F consists of all the edges of G^* joining pairs of vertices of U.

Definition 2.13 A graph of order n with all possible edges is called a complete graph of order n and is denoted by K_n.

Definition 2.14 Let S be a finite semigroup and let T be a non-empty subset of S. The Cayley graph $Cay(S, T)$ of S relative to T is defined as the graph with vertex set S and edge set $\{(x, y) : tx = y \text{ for some } t \in T, x \neq y\}$.

3. Main Results

Throughout this section S is considered as a Factorizable inverse semigroup on a finite set X and G be the group elements, the subgroup of S. Let $N \subseteq K \subseteq S$, where N be the set of all nilpotent elements with nilpotency level 2 and K be the set of all charts or partial symmetries of the form $\begin{pmatrix} i \\ j \end{pmatrix}$,where $i, j \in X$.

Lemma 3.1 Let S be a factorizable inverse semigroup on a finite set X with $x, y \in S$ and $g \in G$. Then xRy if and only if $y = xg$.

Proof. Suppose that xRy. Then $x, y \in R_e$ for some $e \in E$. Therefore, by Theorem2.7, there exist $g_1, g_2 \in G$ such that $x = eg_1$ and $y = eg_2$. From $x = eg_1$, we have $e = xg_1^{-1}$ and so $y = eg_2$ implies $y = (xg_1^{-1})g_2 = x(g_1^{-1}g_2) = xg_3$ for some $g_3 = g_1^{-1}g_2 \in G$.

Conversely suppose that $y = xg$ for some $g \in G$. Therefore $x = yg^{-1}$. Hence xRy. Thus xRy if and only if $y = xg$.

Proposition 3.2 Let S be a factorizable inverse semigroup on a finite set X with $g \in G$ and $x, y \in K$ with $x \neq y$. Then there is an edge between x and y in $Cay(K, N)$ if and only if xLy.

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Proof. Let $x, y \in K$ with $x \neq y$. Suppose that there is an edge between x and y in $Cay(K, N)$. Then there exist $g_1, g_2 \in G$ such that $y = g_1 x$ and $x = g_2 y$. Then by lemma 2.4, we have xLy.

Conversely, suppose that $x, y \in K$ with xLy. Then $x, y \in L_e$ for some $e \in E$. Then by Theorem 2.7, there exist $g_1, g_2 \in G$ such that $x = g_1 e$ and $y = g_2 e$. Then $e = g_1^{-1} x$ and $y = g_2 g_1^{-1} x$. That is $y = g_3 x$ for some $g_3 = g_2 g_1^{-1} \in G$. Since $x \in K$, we have $x = k_1 x$ for some $k_1 = \begin{pmatrix} i \\ i \end{pmatrix} \in K$. Thus $y = g_3 x$ implies $y = (g_3 k_1) x$. Suppose $(g_3 k_1)^2 \neq 0$ (where 0-empty mapping). Then $g_3 k_1 = k_1$ for some $k_1 \in K$,it contradicts our assumption that xLy. Thus $g_3 k_1 \in N$. Hence there is an edge from x to y in $Cay(K, N)$. Also, we have $x = k_1^{-1} g_3^{-1} y = k_1 g_3^{-1} y$ .Therefore there is an edge from y to x. Hence there is an edge between x and y in $Cay(K, N)$.

Proposition 3.3 Let S be a factorizable inverse semigroup on a finite set X and L_e be any $L-$class containing e , where $e \in E(K)$. Then the graph induced by L_e of $Cay(K, N)$ is a complete graph K_n , where $n = |L_e|$.

Proof. Let $x, y \in L_e$ with $x \neq y$. Then we have xLy. Hence, by Proposition 3.2, there is an edge between x and y in $Cay(K, N)$. Hence there is an edge between x and y in the graph induced by L_e of $Cay(K, N)$. Therefore the graph induced by L_e of $Cay(K, N)$ is a complete graph K_n where $n = |L_e|$.

Theorem 3.4 Let S be a factorizable inverse semigroup on a finite set X and L_e be any $L-$class containing e , where $e \in E(K)$. Then $Cay(K, N)$ is the disjoint union of induced subgraphs with vertex set L_e of $Cay(K, N)$.

Proof. Let $e, f \in E(K)$. For $e \neq f$, we have $L_e \cap L_f = \phi$,since they are distinct $L-$classes. Hence the induced subgraphs with vertex set L_e and L_f of $Cay(K, N)$ are disjoint whenever e and f are distinct.

Let us consider $Cay(L_e, N)$ be the graph induced by the vertex set of L_e of $Cay(K, N)$.

Now; $V(Cay(K, N))$ = K = $\bigcup_{e \in E(K)} L_e$ = $\bigcup_{e \in E(K)} V(Cay(L_e, N))$. Let $x, y \in K$ with $x \neq y$. Suppose that there is an edge from x to y in $Cay(K, N)$. Then by Proposition 3.2 ,we have $x\,Ly$. Then $x, y \in L_e$ for some $e \in E(K)$. Since $Cay(L_e, N)$ is complete by Proposition 3.3,there is an edge from x

to y in $Cay(L_e, N)$. Hence there is an edge from x to y in $\bigcup_{e\in E(K)} Cay(L_e, N)$. Therefore $E(Cay(K,N)) \subseteq E(\bigcup_{e\in E(K)} Cay(L_e, N))$. On the other hand , suppose that there is an edge from x to y in the disjoint union of $Cay(L_e, N)$. Since the union is disjoint and each $Cay(L_e, N)$ is complete, it follows that an x and y belongs to
an L_e for some $e \in E(K)$. Hence xLy. Therefore by Proposition 3.2, there is an edge from x to y in $Cay(K,N)$. Hence $E(\bigcup_{e\in E(K)} Cay(L_e, N)) \subseteq$ $E(Cay(K,N))$. Thus $E(Cay(K,N)) =$ $E(\bigcup_{e\in E(K)} Cay(L_e, N))$. Hence $Cay(K,N)$ is the disjoint union of $Cay(L_e, N)$.

Theorem 3.5 Let S be a factorizable inverse semigroup on a finite set X and factorizable as $S = GE$ .If R_e is any R-class of S for $e \in E$, then for $x, y \in S$ with $x \neq y$ there is an edge from x to y in $Cay(S, R_e)$ if and only if there exist an $r \in R_e$ such that $r'xLy$ for every $r' \in R_e \cap L_r$.

Proof. Let $x, y \in S$ with $x \neq y$ .Suppose that there is an edge from x to y in $Cay(S, R_e)$, where R_e is any R-class in S. Then there exist an $r \in R_e$ such that $rx = y$.
Case (1): Let $e = 1$. Then $Cay(S, R_1)$ = $Cay(S, G)$ and so $rx = y$ for some $r \in R_e$ implies $gx = y$ for some $g \in G$. Since $g \in G$, we have $g = g_1 g_2$ for some $g_1, g_2 \in G$. Thus $gx = y$ implies $g_1(g_2 x) = y$ and so $g_2 x = g_1^{-1} y$. Hence $g_2 x L y$ for every $g_2 \in G$.

Conversely suppose that $g_2 x L y$ for every $g_2 \in G$. Then $g_2 x, y \in L_e$ for some $e \in E$. Therefore by Theorem 2.7, there exist g' and $g'' \in G$ such that $g_2 x = g'e$ and $y = g''e$. Then $e = (g')^{-1} g_2 x$ and $y = g''(g')^{-1} g_2 x = g_3 x$ for some $g_3 = g''(g_1)^{-1} g_2 \in G$. Thus there exist an edge from x to y in $Cay(S, G)$.
Case (2): Let $e \neq 1$. Suppose that there is an edge from x to y in $Cay(S, R_e)$. Then there exist an $r \in R_e$ such that $y = rx$. Let $r' \in R_e \cap L_r$. Then $r' \in R_e$ and $r' \in L_r$. Since $r', r \in L_r$, we have $r L r'$. Then $r, r' \in L_e$ for some $e \in E$ .Then by Theorem 2.7, there exist $g_1, g_2 \in G$ such that $r = g_1 e$ and $r' = g_2 e$. Then $e = g_1^{-1} r$ and $r' = g_2 g_1^{-1} r$. That is $r' = g_3 r$ for some $g_3 = g_2 g_1^{-1} \in G$. Therefore $r'x = (g_3 r)x = g_3(rx) = g_3 y$. Thus $r'xLy$ for every $r' \in R_e \cap L_r$.

Conversely suppose that $r'xLy$ for every $r' \in R_e \cap L_r$. Then $r'x, y \in L_e$ for some $e \in E$. Therefore by Theorem 2.7, there exist $g_1, g_2 \in G$ such that $r'x = g_1 e$ and $y = g_2 e$. Then $e = g_1^{-1}(r'x)$ and $y = g_2 g_1^{-1}(r'x)$. That is $y = g_3(r'x)$ for some $g_3 = g_2 g_1^{-1} \in G$. Let $g_3 r' = k'$. Thus $y = k'x$. Since

$r' \in R_e \cap L_r$, we have $r' = g_1 r$ and $r = g_2 r'$ for some $g_2, g_3 \in G$. We can choose an element $g_3 \in G$ such that $r' = g_3 r$ and $r = g_3 r'$. Suppose not, it contradicts $r' \in L_r$. Thus $k' = r \in R_e$ and so there exist an edge from x to y in $Cay(S, R_e)$.

Example 3.6 Let $S = \{e, g, (1,1), (1,2), (2,1), (2,2)\}$ be a set with multiplication defined by

.	e	g	(1,1)	(1,2)	(2,1)	(2,2)
e	e	g	(1,1)	(1,2)	(2,1)	(2,2)
g	g	e	(2,1)	(2,2)	(1,1)	(1,2)
(1,1)	(1,1)	(1,2)	(1,1)	(1,2)	0	0
(1,2)	(1,2)	(1,1)	0	0	(1,1)	(1,2)
(2,1)	(2,1)	(2,2)	(2,1)	(2,2)	0	0
(2,2)	(2,2)	(2,1)	0	0	(2,1)	(2,2)

Then S is a factorizable inverse semigroup on a finite set $X = \{1,2\}$ with group of units $G = \{e, g\}$. Again the $L-$ Classes of S are $L_1 = \{(1,1), (2,1)\}$ and $L_2 = \{(1,2), (2,2)\}$ and the $R-$ Classes of S are $R_1 = \{(1,1), (1,2)\}$ and $R_2 = \{(2,1), (2,2)\}$ respectively.

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