DEGREE SEQUENCE POLYNOMIAL OF GRAPHS

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Abstract

Let $G = (V,E)$ be a graph having order *n* and $< d_1, d_2, ..., d_n >$ be the degree sequence of *G*, where $d_1 \geq d_2 \geq ... \geq d_n$, then the polynomial $\Phi_G(x) = \prod_{i=1}^{n}(x - d_i)$ is called the degree sequence polynomial of *G*. In this paper we give the degree sequence polynomial of some special type of graphs and describe the degree sequence polynomial of the Join, Corona, and Cartesian product of two graphs.

Keywords Degree sequence polynomial of graph, graphic polynomial, polynomial index associated with a positive integer.

Introduction

By a graph we mean a finite undirected graph with neither loops nor multiple edges. The vertex set and the edge set of the graph *G* are denoted by *V*(*G*) and *E*(*G*) respectively. The degree of a vertex $v \in V(G)$, denoted by $\deg_G(v)$, is equal to the cardinality of the neighborhood of v in *G*. For various graph theoretic notations and terminology we follow F. Harrary [1] and for basic number theoretic results we refer [4].

Definition 1.1 Let *G* be a graph of order *n* and $d_1, d_2, ..., d_n$ are degrees of the vertices of *G*, where $d_1 \geq d_2 \geq ... \geq d_n$. Then we refer to the sequence $< d_1, d_2, ..., d_n >$ as the degree sequence of *G*. Then the polynomial $\Phi_G(x)$ given by $\Phi_G(x) = \prod_{i=1}^{n}(x - d_i)$ is called degree sequence polynomial of *G*. A polynomial $f(x)$ is said to be a graphic polynomial if there exist a simple graph *G* such that $\Phi_G(x) = f(x)$.

Every graph *G* has a degree sequence polynomial, but it is not true that a general polynomial is always a degree sequence polynomial of some simple graph *G*. (There is no simple graph *G* with $\Phi_G(x) = x - 1$).

The following are some simple observations which follow immediately from the definition of a degree sequence polynomial.

Observation 1.2 The degree sequence polynomial of some graphs are given below.

- For a complete bipartite graph K_{nm}, $\Phi_{K_{nm}}(x) = (x-m)^n (x-n)^m$
- For the Star graph $K_{1n}, \Phi_{K_{1n}}(x) = (x-n)(x-1)^n$
- For a complete graph K_n, $\Phi_{K_n}(x) = (x-n+1)^n$
- For a path P_n, $\Phi_{P_n}(x) = (x-1)^2 (x-2)^{n-2}$
- For a cycle C_n, $\Phi_{C_n}(x) = (x-2)^n$
- For a wheel W_{n+1}, $\Phi_{W_{n+1}}(x) = (x-3)^n (x-n)$
- For a *k*-regular graph *G* of order *n*, $\Phi_G(x) = (x-k)^n$
- For a Dodecahedron , $\Phi_G(x) = (x-3)^{20}$
- For a Petersen graph , $\Phi_G(x) = (x-3)^{10}$

Observation 1.3 Let G' be a spanning sub graph of G then degree of $\Phi_{G'}(x)$ is less than or equal to the degree of $\Phi_G(x)$

Observation 1.4 The polynomial $f(x)=a_1+a_2x+...+x^n$ is graphic $\Rightarrow a_2$ is even.

In the following remark we present some properties of a degree sequence polynomial.

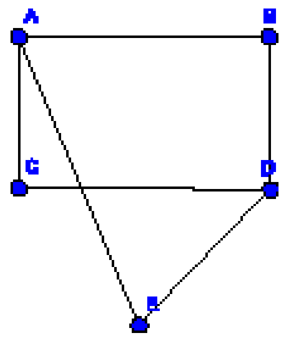
Remark 1.5

1. The degree of the polynomial is the number of vertices of the graph.
2. The absolute value of the coefficient of the second highest degree term of the polynomial is twice the number of edges of the graph.
3. The constant term of the polynomial is equal to the product of the degrees of vertices of the graph.
4. A graph has *n* vertices then every zeros of the polynomial is less than *n*.
5. A graph has pendant vertices then the polynomial has a factor $(x-1)^r$, for some integer $r>0$.
6. For an empty graph (without edges) with *n* vertices the polynomial is x^n.
7. For a null graph the polynomial is *x*.
8. If the degree sequence polynomial has $(x-d)$ as a factor with multiplicity *r* and d is odd then *r* must be even.

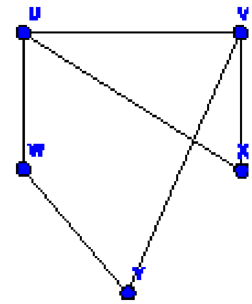
Note 1.6:

1. Isomorphic graphs have same degree sequence polynomials. But the converse need not be true.

   For example the degree sequences of *G* and *H* (see the figure below) are <3,3,2,2,2> and <3,3,2,2,2> respectively, hence the corresponding degree sequence polynomials are $\Phi_G(x)=(x-3)^2(x-2)^3, \Phi_H(x)=(x-3)^2(x-2)^3$. Both are same. But *G* and *H* are not isomorphic.



Graph G



Graph H

2. Consider a graphic polynomial $f(x)$ having a constant term k. There exists infinite number of simple graphs for each $k \geq 1$

For example, if $k=1$ there are simple graphs with degree sequence polynomials are $x^s(x-1)^{2r}, \ s \geq 0, \ r \geq 1$

Similarly for $k=2$ there are simple graphs with degree sequence polynomials are $x^s(x-1)^{2r}(x-2), \ s \geq 0, \ r \geq 1$

Thus, there arises naturally the problem of finding the number of connected simple graphs for a particular value of *k*. So let us define the following.

Definition 1.7: Let $k \geq 1$ be a positive integer. Then the number of connected simple graphs with a graphic polynomial having constant term k is called the polynomial index associated with k and is denoted by $\rho(k)$

Example 1.8: $\rho(1)=1$ since for a connected simple graph the possible degree sequence polynomial is $\Phi_G(x)=(x-1)^2$. Also, $\rho(2)=1$ and $\rho(3)=1$.

But $\rho(4)=2$ since the possible degree sequence polynomials are $(x-1)^4(x-4)$, $(x-1)^2(x-2)^2$

An exclusive study of polynomial index associated with a particular positive integer *k* would provide scope for an independent direction of research, which we leave open at this stage.

1. Main Results

Theorem 2.1 : Let *G* be a graph of order *n*, with degree sequence $< d_1, d_2, ..., d_n >$, then

$\Phi_{G^c}(x) = (-1)^n \Phi_G(n-1-x)$ where G^c is the complement of *G*.

Proof : Let *G* be a graph of order *n*, $v_{1,} v_{2,...,} v_n$ be vertices of *G* such that $d(v_i) = d_i$.

We know that $G \cup G^c = K_n$,

$\deg_G(v_i) + \deg_{G^c}(v_i) = n-1$ so that , $\deg_{G^c}(v_i) = n-1-d_i$

$$\therefore \quad \Phi_{G^c}(x) = \prod_{i=1}^{n}(x-(n-1-d_i))$$

$$= \prod_{i=1}^{n}(x-n+1+d_i)$$

$$= (-1)^n \prod_{i=1}^{n}(-x+n-1-d_i)$$

$$= (-1)^n \Phi_G(n-1-x)$$

Hence the proof.

Theorem 2.2 : Let *G* be a graph of order *n* and *H* be a graph of order *m*. If

V(*G*) $\cap$ V(*H*)= ϕ then $\Phi_{G \cup H}(x) = \Phi_G(x)\Phi_H(x)$.

Proof : Let *G* has *n* vertices having degree sequence $<d_1, d_2, ..., d_n>$ and *H* has *m* vertices with degree sequence $<p_1, p_2, ..., p_m>$. $G \bigcup H$ has mn vertices having degrees $d_1, d_2, ..., d_n, p_1, p_2, ..., p_m$. Hence $\Phi_{G \bigcup H}(x) = \prod_{i=1}^{n}(x-d_i)\prod_{i=1}^{m}(x-p_i) = \Phi_G(x)\Phi_H(x)$.

Definition 2.3: Join of graphs

The Join *G* +*H* of two graphs *G* and *H* is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and the edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$

Theorem 2.4: Let *G* and *H* be graphs with degree sequence polynomials $\Phi_G(x)$ and $\Phi_H(x)$ having orders *n*, *m* respectively. Then the degree sequence polynomial of *G* +*H* is

$$\Phi_{G+H}(x) = \Phi_G(x-m)\Phi_H(x-n)$$

Proof : Let *G* has *n* vertices and $<d_1, d_2, ..., d_n>$ be the degree sequence of *G*. Let *H* has *m* vertices and $<p_1, p_2, ..., p_m>$ be the degree sequence of *H*.

Let $v \in V(G+H)$ = V(*G*) $\bigcup$ V(*H*)

If $v \in V(G)$ then the neighborhood of *v* in $V(G+H)$ = the neighborhood of *v* in $V(G)$ + m

If $v \in V(H)$ then

the neighborhood of *v* in $V(G+H)$ = the neighborhood of *v* in $V(H)$ + n

$\therefore$ Degree sequence of vertices in $G+H$ are $d_1+m, d_2+m, ..., d_n+m, p_1+n, p_2+n, ..., p_m+n$ in some order. It follows that

$$\Phi_{G+H}(x) = \prod_{i=1}^{n}(x-(d_i+m))\prod_{j=1}^{m}(x-(p_j+n))$$

$$=\prod_{i=1}^{n}(x-m-d_i)\prod_{j=1}^{m}(x-n-p_j)$$

$$= \Phi_G(x-m)\Phi_H(x-n)$$

Hence the proof.

Corollary 2.5: Let *G* be a graph with polynomial $\Phi_G(x)$ has order *n*, and *H* be a k-regular graph of order *m* . The polynomial $\Phi_{G+H}(x)$ is $\Phi_{G+H}(x)=(x-(n+k))^n\Phi_G(x-m)$

Proof : Using Theorem 2.1 and *H* is k-regular graph of order *m* so

$\Phi_H(x-n)=(x-n-k)^m=(x-(n+k))^m$.

$\therefore \quad \Phi_{G+H}(x)=\Phi_G(x-m)\Phi_H(x-n)=(x-(n+k))^n\Phi_G(x-m)$.

Corollary 2.6: Let *n* and *m* are positive integer . Then $\Phi_{K_{mn}}(x)=(x-m)^n(x-n)^m$

Proof : *G* be an empty graph(without edge) with *n* vertices and *H* be an empty graph(without edge) with *m* vertices . Then $G+H=K_{mn}$ and $\Phi_G(x)=x^n,\Phi_H(x)=x^m$

Using Theorem 2.1, we get

$\Phi_{K_{mn}}(x)=\Phi_{G+H}(x)=\Phi_G(x-m)\Phi_H(x-n)=(x-m)^n(x-n)^m$.

Definition 2.7: Corona of graphs

The corona $G\circ H$ of two graphs *G* and *H* is the graph obtained by taking one copy of *G* of order *n* and *n* copies of *H*, and then joining the ith vertex of *G* to every vertex in the ith copy of *H*. For every $v\in V(G)$, denote H^v the copy of *H* whose vertices are attached one by one to the vertex *v*. Subsequently denoted by $v+H^v$ the sub graph of the corona $G\circ H$ corresponding to the join <{*v*}>+H^v, $v\in V(G)$

Theorem 2.8: Let *G* be a connected graph with order *n* and *H* any graph of order m. Then

$\Phi_{G\circ H}(x) = \Phi_G(x-m)(f_H(x-1))^n$

Proof : Let *G* has *n* vertices with degree sequence $< d_1, d_2, ..., d_n >$ and *H* has m vertices with degree sequence $< p_1, p_2, ..., p_m >$.

Let $v \in V(G \circ H)$

If $v \in V(G)$ then the neighborhood of *v* in $V(G+H)$ = the neighborhood of *v* in $V(G)$ + number of vertices in *H* = the neighborhood of *v* in $V(G)$ +m

If $v \in V(H)$ then this *v* has *n* copies and each has one more degree in $G \circ H$ as in *H.*

Hence the neighborhood of *v* in $V(G \circ H)$ = the neighborhood of *v* in $V(H)$ +1

$\therefore$ The degree sequences of vertices in $G+H$ are

$d_1+m, d_2+m, ..., d_n+m, p_1+1, p_2+1, ..., p_m+1$ in some order, each p_i+1 with multiplicity *n* for $1 \le i \le m$. It follows that

$$\Phi_{G\circ H}(x) = \prod_{i=1}^{n}(x-(d_i+m))\left[\prod_{j=1}^{m}(x-(p_j+1))\right]^n$$

$$\prod_{i=1}^{n}(x-m-d_i)\left[\prod_{j=1}^{m}(x-1-p_j)\right]^n$$

$$= \Phi_G(x-m)\left[\Phi_H(x-1)\right]^n$$

Hence the proof.

Corollary 2.9: Let *G* be a connected graph of order *n* and *H* an *r*-regular graph of order *m*, then

$\Phi_{G\circ H}(x) = ((x-1-r)^m)^n \Phi_G(x-m) = (x-(r+1))^{mn}\Phi_G(x-m)$.

Vol.09 Issue-01, (January - June, 2017) ISSN: 2394-9309 (E) / 0975-7139 (P)
Aryabhatta Journal of Mathematics and Informatics (Impact Factor- 5.856)

Corollary 2.10: Let *G* be a connected graph of order *n* and *H* a complete graph of order *m*, then

$\Phi_{G\circ H}(x)=(x-m)^{mn}\Phi_G(x-m)$.

Definition 2.11: Cartesian product of graphs

The Cartesian product $G\times H$ of two graphs G and H is the graph with $V(G\times H)=V(G)\times V(H)$ and $(u,u^{'})(v,v^{'})\in E(G\times H)$ if and only if either $uv\in E(G)$ and $u^{'}=v^{'}$ or $u^{'}v^{'}\in E(H)$ and $u=v$.

Theorem 2.12: Let *G* be a graph having *n* vertices, $<d_1,d_2,...,d_n>$ be the degree sequence and *H* has *m* vertices wit degree sequence $<p_1,p_2,...,p_m>$. Then

$$\Phi_{G\times H}(x)=\prod_{i=1}^{m}\Phi_G(x-p_i)=\prod_{i=1}^{n}\Phi_H(x-d_i)$$

Proof : Let *G* has *n* vertices $v_1,v_2,...,v_n$, $<d_1,d_2,...,d_n>$ be the degree sequence of *G* such that $\deg_G(v_i)=d_i$ and *H* has *m* vertices $u_1,u_2,...,u_n$, $<p_1,p_2,...,p_m>$ be its degree sequence such that $\deg_H(u_i)=p_i$.

Let $(u_iv_j)\in V(G\times H)$ the degree of (u_iv_j) is d_i+p_j for $1\le i\le n, 1\le j\le m$

$\therefore$ The degree sequences of vertices in $G\times H$ are

$d_1+p_1,d_1+p_2,...,d_1+p_m,d_2+p_1,d_2+p_2,...,d_2+p_m,.,..,d_n+p_1...,d_n+p_m$ in some order.

It follows that

$$\Phi_{G\times H}(x)=\prod_{i=1}^{n}(x-(d_i+p_1))\prod_{i=1}^{n}(x-(d_i+p_2))...\prod_{i=1}^{n}(x-(d_i+p_m))$$

$$=\prod_{i=1}^{n}(x-d_i-p_1)\prod_{i=1}^{n}(x-d_i-p_2)...\prod_{i=1}^{n}(x-d_i-p_m)$$

$$=\Phi_G(x-p_1)\Phi_G(x-p_2)\ldots\Phi_G(x-p_m)$$

$$=\prod_{i=1}^{m}\Phi_G(x-p_i)$$

Also,

$$\Phi_{G\times H}(x)=\prod_{i=1}^{m}(x-(d_1+p_i))\prod_{i=1}^{m}(x-(d_2+p_i))\ldots\prod_{i=1}^{m}(x-(d_n+p_i))$$

$$=\prod_{i=1}^{m}(x-p_i-d_1)\prod_{i=1}^{m}(x-p_i-d_2)\ldots\prod_{i=1}^{m}(x-p_i-d_n)$$

$$=\Phi_H(x-d_1)\Phi_H(x-d_2)\ldots\Phi_H(x-d_n)=\prod_{i=1}^{n}\Phi_H(x-d_i).$$

Corollary 2.13: Let *G* has *n* vertices, $<d_1,d_2,\ldots,d_n>$ be degree sequence and *H* is cycle of order *m* . Then

$$\Phi_{G\times H}(x)=\left[\Phi_G(x-2)\right]^m$$

Proof : $\Phi_H(x)=(x-2)^m$ using Theorem 2.12

$$\Phi_{G\times H}(x)=\prod_{i=1}^{n}\Phi_H(x-d_i)=\prod_{i=1}^{n}\left[(x-d_i-2)^m\right]=\left[\prod_{i=1}^{n}(x-d_i-2)\right]^m$$

$$=\left[\Phi_G(x-2)\right]^m$$

Corollary 2.14: Let *G* has *n* vertices, $<d_1,d_2,\ldots,d_n>$ be its degree sequence and *H* is *k*-regular graph of order *m* . Then

$$\Phi_{G\times H}(x)=\left[\Phi_G(x-k)\right]^m$$

Proof : Using Theorem 2.12 , each $p_i=$ k

$$\Phi_{G\times H}(x)=\prod_{i=1}^{m}\Phi_G(x-p_i)=\prod_{i=1}^{m}\Phi_G(x-k)$$

$$=\left[\Phi_G(x-k)\right]^m .$$

Corollary 2.15: Let *G* has *n* vertices, $<d_1,d_2,...,d_n>$ be its degree sequence and $H=K_{pq}$. Then

$$\Phi_{G\times H}(x)=\left[\Phi_G(x-p)\right]^q\left[\Phi_G(x-q)\right]^p$$

Proof : K_{pq} has *p +q* number of vertices and *p* of them are degree *q* and *q* of them are degree *p* .

Then using Theorem 2.12, we get

$$\Phi_{G\times H}(x)=\prod_{i=1}^{p+q}\Phi_G(x-p_i)=\prod_{i=1}^{p}\Phi_G(x-q)\prod_{i=1}^{q}\Phi_G(x-p)$$

$$=\left[\Phi_G(x-p)\right]^q\left[\Phi_G(x-q)\right]^p .$$

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