k-Product Graph Labeling

Asif Navas KR,Research Scholar,
kannur University, kannur- 670 567,
kerala, india

Dr Ajitha V,Department of Mathematics,
M. G. College ,Iritty-670703,
Kerala, India

Abstract
A (p, q) graph is said to admit k-Product labeling if its vertex can be labeled by positive integers such that the induced edge labels obtained by the absolute difference of the labels of end vertices are the first k-Product numbers. A graph G which admit k-Product labeling is called k-Product graph. In this paper, we prove path admits k-Product labeling and combs, stars, bistars, coconut trees admit 2-Product labeling and also cycle C_n is not 2-Product when n is not congruent to 0 mod 4. Also C_n admits 2-Product labeling iff the set of all first n 2-Product numbers is a zero sum set.

Keywords k-Product labeling, 2-Product labeling, zero sum set.

Introduction

All graphs in this paper are finite, simple and undirected. The vertex set and edge set of graph G are denoted by V(G) and E(G) respectively. For various graph theoretic notations and terminology we follow Harry [1] and for number theory we follow Burton [2]. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. A dynamic survey on graph labeling is regularly updated by Gallian [3] and it is published by Electronic Journal of Combinatorics. Vast amount of literature is available on different types of graph labeling and more than 1000 research papers have been published so far in last four decades.

Definition 1.1

A k-Product number is a number obtained by multiplying k-consecutive natural numbers (starting from 2) less than or equal to a given positive integer n. If the n^{th} k-Product number is denoted by

$$A_n^{(k)} = (\mathrm{n}+1)(\mathrm{n}+2)\ldots.(\mathrm{n}+\mathrm{k})$$

The 2-Product numbers are 6, 12, 20, 30, 42, 56, 72, 90, 110...............

Definition 1.2

A k-Product labeling of a graph G is a one-one function $f: V(G) \to \mathbb{Z}^+$ that induces a bijection $f^+: E(G) \to \{A_1^{(k)}, A_2^{(k)}, \ldots\ldots, A_q^{(k)}\}$ of the edges defined by $f^+(uv) = |f(u) - f(v)|\ \forall\ e = uv \in$ E(G). The graph which admit such labeling is called a k-Product graph.

Definition 1.3

A finite subset F_n of positive integers is said to be zero sum set if there exist a linear combination, $\sum_{i=1}^{n} a_i x_i = 0$, $x_i \in F_n$ for $i = 1,2,\ldots,n$ and $a_i \in \{-1,1\}$ for $i = 1,2,\ldots,n$.

Definition 1.4
A 2-Product number is a number obtained by multiplying 2-consecutive natural numbers (starting from 2) less than or equal to a given positive integer n. If the n^{th} 2-Product number is denoted by

$$A_n^{(2)} = (\mathrm{n}+1)(\mathrm{n}+2)$$

The 2-Product numbers are 6, 12, 20, 30, 42, 56, 72, 90, 110..............

Definition 1.5
The n^{th} sum of k-Product numbers is denoted by $S_{A_n^{(k)}}$,

$$S_{A_n^{(k)}} = \sum_{i=1}^{n} A_i$$

Main Results

Here we prove that combs, stars, bistars, coconut trees admits 2-Product graph labeling. Also, cycle C_n is not 2-Product graph if n is not congruent to $0(mod4)$. And C_n admits 2-Product labeling iff the set of all first n 2-Product numbers is a zero sum set.

Theorem 2.1
The path P_n admits k-Product labeling.
Proof:
Let $P_n : u_1, u_2, \ldots\ldots, u_n$ be the path and let $w_i = u_i u_{i+1}$ ($1 \leq i \leq n$) be the edges.
for $i = 1,2,\ldots,n$, defined

$$f(u_i) = \frac{\mathrm{i}(\mathrm{i}+1)(\mathrm{i}+2)\ldots.(\mathrm{i}+\mathrm{k})}{k+1}$$

We will show that the induced edge labels obtained by the absolute difference of end vertices are the first $(n-1)$ k-Product numbers.

For $1 \leq i \leq n-1$,

$$\begin{aligned}|f(u_i) - f(u_{i+1})| &= \left|\frac{\mathrm{i}(\mathrm{i}+1)(\mathrm{i}+2)\ldots.(\mathrm{i}+\mathrm{k})}{k+1} - \frac{(\mathrm{i}+1)(\mathrm{i}+2)\ldots.(\mathrm{i}+\mathrm{k})(\mathrm{i}+\mathrm{k}+1)}{k+1}\right| \\ &= \left|\frac{(\mathrm{i}+1)(\mathrm{i}+2)\ldots.(\mathrm{i}+\mathrm{k})[\mathrm{i}-(\mathrm{i}+\mathrm{k}+1)]}{k+1}\right| \\ &= (\mathrm{i}+1)(\mathrm{i}+2)\ldots.(\mathrm{i}+\mathrm{k}) \\ &= A_i^{(k)} \\ &= f^i(w_i)\end{aligned}$$

Thus the induced edge labels are the first $n-1$ k-Product numbers.

Thus path P_n admits 2-product labeling .

Example 2.2
The 2-product labeling of P_6 is shown below.

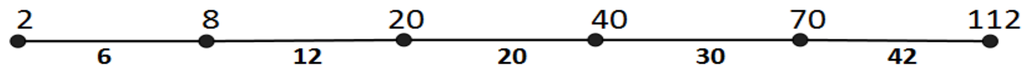

Figure 1

Theorem 2.3
The comb $P_n©K_1$ admits 2-Product labeling.
Proof:
Let $P_n: u_1, u_2, \ldots.., u_n$ be the path and let $w_i = u_i u_{i+1}$ $(1 \le i \le n-1)$ be the edges. Let $v_1, v_2, \ldots.., v_n$ be the pendant vertices adjacent to $u_1, u_2, \ldots.., u_n$ respectively and $t_i = u_i w_i$ $(1 \le i \le n)$ be the edges.

for $i = 1,2, \ldots, n,$ define

$$f(u_i) = \frac{\mathrm{i(i+1)(i+2)}}{3}$$
$$f(v_i) = \frac{3(n^2 + 2ni + n) + 6i^2 + 5i + i^3}{3}$$

Then,

$$|f(u_i) - f(u_{i+1})| = f^+(w_i) \quad for\ 1 \le i \le n-1$$
$$|f(u_i) - f(v\_i)| = f^i(t_i) \quad for\ 1 \le i \le n$$

Thus the induced edge labels are the first $(2n-1)$ 2-Product numbers.
Hence comb $P_n©K_1$ admits 2-Product labeling.

Example 2.4
The 2-Product labeling of $P_5©K_1$ is shown below.

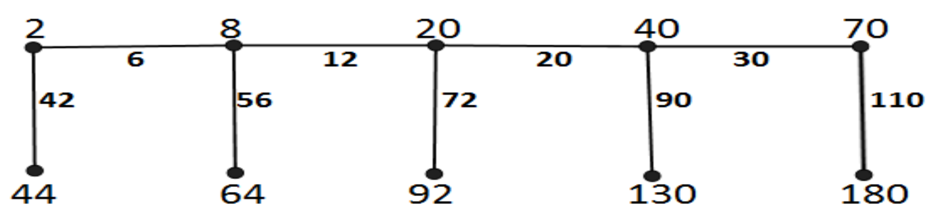

Figure 2

Theorem 2.5
The star graph $K_{1,n}$ admits 2-Product labeling.
Proof:
Let u be the apex vertex and let $v_1, v_2, \ldots\ldots, v_n$ be the pendant vertices of the star $K_{1,n}$.
Define,

$$f(u) = 1$$
$$f(u_i) = (i+1)(i+2)+1 \quad for\ 1 \leq i \leq n-1$$

We see that induced edge labels are the first n 2-Product numbers.
Hence star graph $K_{1,n}$ admits 2-Product labeling.

Example 2.6
The 2-Product labeling of $K_{1,6}$. is shown below.

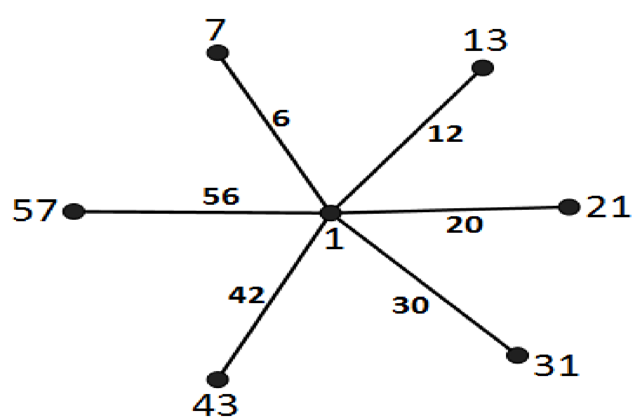

Figure 3

Theorem 2.7
The bistar $B_{m,n}$ admits 2-Product labeling.
Proof:
Let,

$$V(B_{m,n}) = \{u, v, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\},$$
$$E(B_{m,n}) = \{uv, uu_i, vv_j : 1 \leq i \leq m, 1 \leq j \leq n\}$$

Define f by,

$$f(u) = 1$$
$$f(v) = 7$$
$$f(u_i) = (i+1)(i+2)+1 \quad for\ 1 \leq i \leq m$$
$$f(v_j) = (m+2+i)(m+3+i)+7 \quad for\ 1 \leq j \leq n$$

we see that induced edge labels are the first $m + n + 1$ 2-Product numbers.
Hence bistar $B_{m,n}$ admits 2-Product labeling.

Example: 2.8
The 2-Product labeling of $B_{4,3}$ is shown below.

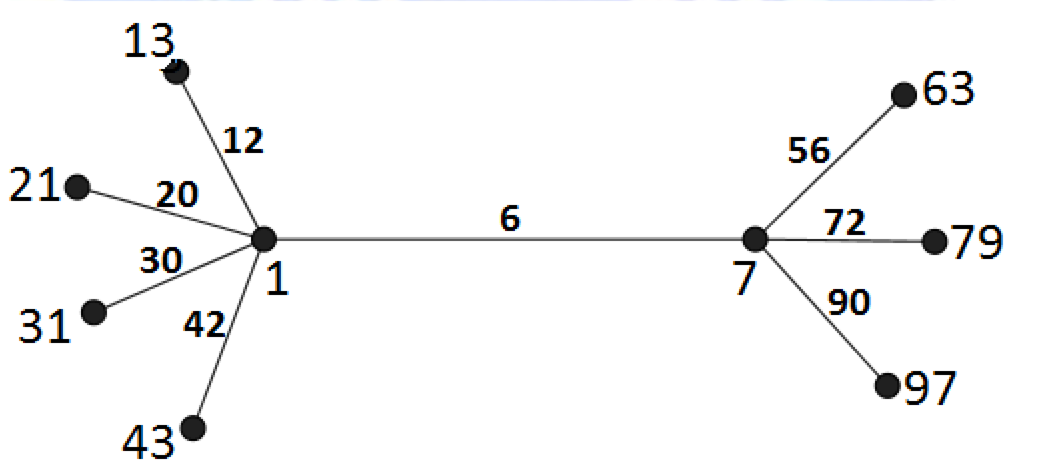

Figure: 4

Theorem: 2.9
Coconut tree admits 2-Product labeling.
Proof:
Let $v_1, v_2, \ldots\ldots, v_n$ be the vertices of a path and $w_1, w_2, \ldots\ldots, w_m$ be the pendant vertices, being adjcent to v_1.
Define f by,

$$f(v_j) = \frac{j(j+1)(j+2)}{3} \quad for\ 1 \le j \le n$$
$$\mathrm{f(w_i) = (n+i)(n+i+1) + 2 \ \ for\ 1 \le i \le m}$$

we see that induced edge labels are the first $m + n - 1$ 2-Product numbers.
Hence Coconut tree admits 2-Product labeling.

Example: 2.10
The 2-Product labeling of coconut tree is shown below.

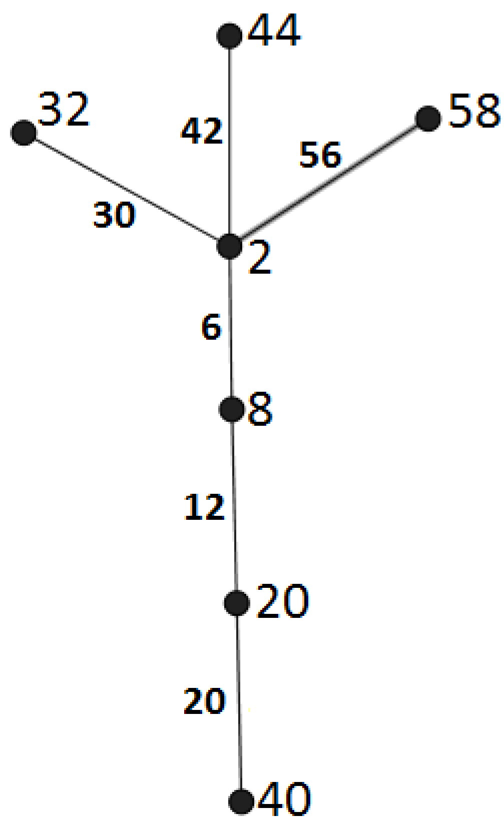

Figure: 5

Result: 2.11

The n^{th} sum of 2-Product numbers is denoted by $S_{A_n^{(2)}}$,

$$S_{A_n^{(2)}} = \frac{n(n+1)(2n+10)+12n}{6}.$$

Theorem: 2.12

If F_n be a zero sum set then S_{F_n} is an even number, where $S_{F_n} = \sum_{i=1}^{n} x_i$, $where\ x_i \in F_n$.

Proof:

Suppose $F_n = \{x_1, x_2, \ldots., x_n\}$ be a zero sum set.

Then ∃ a linear combination, $\sum_{i=1}^{n} a_i x_i$, where $a_i \in \{1, -1\}$, $x_i \in F_n$.

Let $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ be the elements in F for which $a_{i_j} = 1\ for\ j = 1,2, \ldots, k$.

Then,

$$x_{i_1} + x_{i_2} + \cdots + x_{i_k} - x_{i_{k+1}} - x_{i_{(k+1)}} - \cdots . - x_{i_n} = 0$$
$$x_{i_1} + x_{i_2} + \cdots + x_{i_k} = x_{i_{k+1}} + x_{i_{(k+1)}} + \cdots + x_{i_n}$$

Let $\sum_{i=1}^{n} x_i = m$,

Then, $m = 2(x_{i_1} + x_{i_2} + \cdots + x_{i_k})$

Thus, S_{F_n} is an even number.

Theorem: 2.13

An n −cycle C_n admits 2-Product labeling iff $\{A_1^{(2)}, A_2^{(2)}, \ldots\ldots, A_n^{(2)}\}$ is a zero sum set.

Proof:

Suppose C_n admits 2-Product labeling.

Let $C_n : u_1, u_2, \ldots, u_n, u_1$ be the n −cycle.

Let $f(u_i) = x_i$ for $i = 1,2, \ldots, n$.

Then, we have,

$$|x_1 - x_2| = A_{i_1}^{(2)}$$
$$|x_2 - x_3| = A_{i_2}^{(2)}$$
$$\vdots$$
$$|x_n - x_1| = A_{i_n}^{(2)}$$

Where, $A_{i_j}^{(2)} \in \{A_1^{(2)}, A_2^{(2)}, \ldots\ldots, A_n^{(2)}\}$

Above equations can be written as

$$x_1 - x_2 = a_{i_1} A_{i_1}^{(2)} \qquad (1)$$
$$x_2 - x_3 = a_{i_2} A_{i_2}^{(2)} \qquad (2)$$
$$\vdots$$
$$x_n - x_1 = a_{i_n} A_{i_n}^{(2)} \qquad (n)$$

Adding $(1) + (2) + \ldots + (n)$, we get

$$\sum_{j=1}^{n} a_{i_j} A_{i_j}^{(2)} = 0$$

Which implies, $\sum_{i=1}^{n} a_i A_i = 0$

Thus $\left\{A_1^{(2)}, A_2^{(2)}, \ldots\ldots, A_n^{(2)}\right\}$ is a zero sum set.

Conversly, suppose $\left\{A_1^{(2)}, A_2^{(2)}, \ldots\ldots, A_n^{(2)}\right\}$ is a zero sum set. We have to prove n-cycle C_n admits 2-Product labeling.

Since, $\sum_{i=1}^{n} a_i A_i = 0 \quad$ for $a_i \in \{1, -1\}$

Thus , we have,

$$A_{i_1} + A_{i_2} + \cdots + A_{i_k} - A_{i_{k+1}} - A_{i_{(k+1)}} - \cdots . - A_{i_n} = 0$$

Where , $A_{i_j} \in \left\{A_1^{(2)}, A_2^{(2)}, \ldots\ldots, A_n^{(2)}\right\}$
Now, we define ,

$$f(u_1) = 1$$
$$f(u_{j+1}) = f(u_1) + A_{i_j} \quad for\ \mathrm{j} = 1,2, \ldots, \mathrm{k}.$$
$$f(u_{j+1}) = f(u_j) - A_{i_j} \quad for\ \mathrm{j} = \mathrm{k}+1, \ldots, \mathrm{n}.$$

we see that induced edge labels are the first n 2-Product numbers.
Thus C_n is 2-Product graph.

Lemma: 2.14

$\frac{S_{A_n^{(2)}}}{2}$ is an odd number when n is not congruent to 0 $mod(4)$.

Proof:

We have,

$$S_{A_n^{(2)}} = \frac{n(n+1)(2n+10)+12n}{12}$$

For $n = 4k+1$,

$$S_{A_{4k+1}^{(2)}} = \frac{(4k+1)[(4k+2)(8k+12)+12]}{12}$$
$$= 2\left(\frac{(4k+1)(4k+2)(4k+6)+6(4k+1)}{3}\right)$$
$$= \ 2\left(\frac{(4k+1)(4k+2)(4k+6)+24k}{12}\right)+1$$

Now,

$$(4k+1)(4k+2)(4k+6)+24k \equiv (4k+1)(4k+2)(4k+6)\ mod(12)$$
$$\equiv 64\mathrm{k}^3 + 8\mathrm{k}\ mod\ (12)$$

By division algorithm , k is one the form $3s, 3s+1, 3s+2$,
If $k = 3s$,

$$64k^3 + 8k \ = \ 64(3s+1)^3 + 8(3s+1)$$
$$= 64[27s^3 + 27s^2 + 9s + 1] + 24s + 8$$
$$= 12[144s^3 + 144s^2 + 50s + 6]$$

Thus, $64k^3 + 8k \equiv 0\ (mod 12)$
Similarly, if $k = 3s+2$, then $64k^3 + 8k \equiv 0\ (mod 12)$.
Thus,

$$(4k+1)(4k+2)(4k+6)+24k \ \equiv 0\ (mod 12)$$

Thus, $\frac{S_{A_n^{(2)}}}{2}$ is an odd number when $n = 4k+1$.

For $n = 4k+2$,

$$S_{A_{4k+2}^{(2)}} = \frac{(4k+2)[(4k+3)(8k+14)+12]}{12}$$

$$= \left(\frac{(4k+2)(4k+3)(8k+14)+12(4k+1)+12}{12}\right)$$

$$= 2\left(\frac{(2k+1)(4k+3)(8k+14)+6(4k+1)}{12}\right)+1$$

It can be easily verify that $(2k+1)(4k+3)(8k+14)+6(4k+1) \equiv 0\ (mod12)$.

For $n = 4k+3$,

$$S_{A_{4k+3}^{(2)}} = \frac{(4k+3)[(4k+4)(8k+16)+12]}{12}$$

$$= \left(\frac{(4k+4)(4k+3)(8k+16)+12(4k+2)+12}{12}\right)$$

$$= 2\left(\frac{(2k+2)(4k+3)(8k+16)+6(4k+2)}{12}\right)+1$$

It can be easily verify that $(2k+2)(4k+3)(8k+16)+6(4k+2) \equiv 0\ (mod12)$.

Thus, $\frac{S_{A_n^{(2)}}}{2}$ is an odd number when n is not congruent to 0 $mod(4)$.

Theorem: 2.15

A cycle C_n doesn't admits 2-Product when n is not congruent to 0 $mod(4)$.

Proof:

For each $\in \mathbb{N}$, $A_n^{(2)} = (n+1)(n+2)$ is an even number.

By **Lemma 2.14** , $\frac{S_{A_n^{(2)}}}{2}$ is an odd number when n is not congruent to 0 $mod(4)$. Thus $\left\{A_1^{(2)}, A_2^{(2)}, \ldots\ldots, A_n^{(2)}\right\}$ is not a zero sum set when n is not congruent to 0 $mod(4)$.

Then by **Theorem 2.13,**

C_n doesn't admits 2-Product labeling for when n is not congruent to 0 $mod(4)$.

Concluding Remarks

In this paper, we introduce a new concept of k-Product graph labelling, we proved 2-Product graph labelling of standard graphs and introduced zero sum set and its application in labelling of graphs. To investigate analogues result for different graphs and extend the labelling further towards a generalization is an open area of research.

References

[1] F. Harary Graph Theory, Addition-Wesley, Reading, Mass, 1972.

[2] David M. Burton, Elementary Number Theory, (Sixth edition), TATA McGRAW- HILL(2006).

[3] J.A. Gallian, A dynamic survey of graph labelling, Electronic Journal of Combinatorics, 17 (2010), DS6.

[4] S. Murugesan, D. Jayaraman, J. Shiama Some Higher Order Triangular Sum Labeling of Graphs,International Journal of Computer Applications (0975 - 8887) Volume 72 - No. 10, June 2013.

[5] S. Murugesan,D. Jayaraman,J. Shiama Tetrahedral and Pentatopic Sum Labeling of Graphs, International Journal of Applied Information Systems (IJAIS) ISSN : 2249-0868 Foundation of Computer Science FCS, New York, USA Volume 5 - No. 6, April 2013 - www.ijais.org.

[6] S. Arumugam, K. A. Germina, and Ajitha V, On Square Sum graphs, AKCE International Journal of Graphs and Combinatorics, 1-10 (2008).