Vol.09 Issue-01, (January - June, 2017) ISSN: 2394-9309 (E) / 0975-7139 (P)
Aryabhatta Journal of Mathematics and Informatics (Impact Factor- 5.856)

Applications of Fuzzy Subdomains in Fuzzy G-subdomains

A.R. Alizadeh M.

Department of Mathematics, Faculty of Science,

Yasouj University, Yasouj, Iran

Abstract. One of the most important issues in fuzzy logic is the application of fuzzy logic in the field of algebraic structures, because of this issue several articles have been published. In this paper, the important application of fuzzy rings to fuzzy G-subdomains are discussed, on this regard, some sources about fuzzy subrings, fuzzy ideals and fuzzy subdomains have used. According to these researches and by the concepts of the commutative rings which is presented by Kaplansky, some fuzzy G-subdomains properties and related theorems are discussed.

2010 Mathematics Subject Classification. Primary 03E72, 08A72, Secondary 03E75, 20N25

Key words and phrases. G-Domain, Fuzzy G-subdomain, Fuzzy Subset, Fuzzy Subdomain

1. Elementary Concepts of Fuzzy Algebraic Structures

At first some operations on fuzzy subrings in a commutative ring *R* are introduced. In addition the set of all fuzzy subsets on *R* denoted by *F(R).*

Definition 1.1. [3] Let R is a ring and $A \in F(R)$, A is called a fuzzy subring of R if for all $a . b \in R$:
i) $A(a-b) \geq inf\ \{A(a).A(b)\} = (A(a) \wedge A(b))$.
ii) $A(ab) \geq inf\ \{A(a).A(b)\} = (A(a) \wedge A(b))$.

Definition 1.2.*A* is a fuzzy subdomainof *R[x]*if:
i) *A* is a fuzzy subring of *R[x]*.
ii) For all $f.g \in R[x]$ if $A(fg) = 0$ then, $A(f) = 0$ or $A(g) = 0$.
It is obvious that $A(fg) \geq inf\ \{A(f).A(g)\}$, because *A* is also a fuzzy subring of *R[x]*.
The set of all fuzzy subdomains of *R[x]* is denoted by $F_D(R[x])$.

Definition 1.3. For each $A \in F_D(R[x])$, the $A_t = \{f \in R[x] | A(f) \geq t\}$, $\forall t \in [0.1]$ is called *"t-cut"* or *"t- level set"*of *R[x]*.

Theorem 1.1.$A \in F_D(R[x])$ if and only if $\forall t \in [0.1]$.the t-level set of A_t is a subdomain of *R[x]*.

See Theorem.**Proof**"1.2" of [2]. □

Definition.1.4A nonempty fuzzy subset A of R[x] is said to be ideal if and only if , for any $f.g \in R[x]$:

i) $A(f-g) \geq A(f) \wedge A(g)$.
ii) $A(fg) \geq A(f) \vee A(g)$.

Note 1.1.1) If R is a commutative ring then the condition of"ii"in Definition of "1.4" is equivalent to follows:

$$A(fg) \geq A(f) \;.\; \forall f.g \in R[x]$$

2) The set of all fuzzy ideals on R[x] is denoted by $F_I(R[x])$.

Theorem 1.2. Let $A \in F_D(R[x])$, then $A \in F_I(R[x])$ if and only if A_t is an ideal of R[x], $\forall t \in \{A(R[x])\} \cup \{r \in [0.1]: r \leq A(R[0])\}$.
Where, $A(R[x]) = \{A(f): f \in R[x]\}$.

Proof. Let $A \in F_I(R[x])$ and $t \in [0.1]$ be such that:

$$t \leq A(R[0]) \quad or \quad t \in \{A(R[x])\}$$

If $f.g \in A_t$ and $h \in R[x]$ then:

$$A(f-g) \geq inf\{A(f).A(g)\} \geq t$$

And

$$A(hf) \geq A(f) \geq t$$

Hence $f-g.hf \in A_t$.Thus A_t is an ideal of $R[x]$.
Conversely, let A_t be an ideal of *R[x]* and$t \in \{A(R[x])\} \cup \{r \in [0.1]: r \leq A(R[0])\}$
Let $f.g.h \in R[x]$ and $t = inf\{A(f).A(g)\}$.Then $t \leq A(R[0])$ and $f.g \in A_t$. Thus:

$$f-g \in A_t$$

Hence:

$$\big(A(f-g)\big) \geq t = inf\{A(f).A(g)\}.$$

Let $s = A(f), A(f) \in A(R[x])$.Then $f \in A_s$.Thus $hf \in A_s$ since A_s is an ideal of $R[x]$.Hence $\big(A(hf)\big) \geq s = A(f)$. Thus A is a fuzzy ideal of $R[x]$. □

2. Fuzzy G-Subdomains

Remark 2.1. [6]Let *D*be an integral domain with quotient field *K*, the following two statements are equivalent:
i) *K* is a finitely generated ring over *D*.
ii) *K*as a ring, can be generated over *D* by one element.

Definition 2.1. An integral domain satisfying either (hence both) of the statements in Remark "2.1" is called an *G-domain*.

The name honors Oscar Goldman. His paper [5] appeared at virtually the same time as a similar paper by Krull [7]. Since Krull already has a class of rings named after him, it seems advisable not to attempt to honor Krull in this connection. Further results concerning the material in this section appear in Gilmer's paper [4].

Definition 2.2. Let *A* be a fuzzy subset of domain *D*, the D_A is a subdomain of *D* if it is generated by the set of $S = \{x \in D : 0 < A(x) \leq 1\}$. (i.e., D_A is the intersection of all subdomains of *D* such that each of them contains the set of *S*). It is obvious that the D_A is the smallest subdomain of *D* in it contains *S*.

Lemma 2.1.[2] Let *D* is a domain, the fuzzy subset *A* is a fuzzy*G-subdomain* of *D* if D_A as a subdomain of *D* itself is a *G-domain*.

Example 2.1. Let $\mathbb{Q}$ be the Rational numbers, since for each prime number *2*, *3*, ... the extended fields $\mathbb{Q}[\sqrt{2}\,]$, $\mathbb{Q}[\sqrt{2}\,.\sqrt{3}].\mathbb{Q}[\sqrt{2}.\sqrt{3}.\sqrt{5}]$ *and* $\mathbb{Q}[\sqrt{2}.\sqrt{3}.\sqrt{5}.\sqrt{7}]$ are *G-domains*. If we define *A(x)* as the following:

$$A(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ & 1/2\, x \in \mathbb{Q}[\sqrt{2}] - \mathbb{Q} \\ 1/3 & x \in \mathbb{Q}[\sqrt{2}.\sqrt{3}] - \mathbb{Q}[\sqrt{2}] \\ 1/5 & x \in \mathbb{Q}[\sqrt{2}.\sqrt{3}.\sqrt{5}] - \mathbb{Q}[\sqrt{2}.\sqrt{3}] \\ 1/7 & x \in \mathbb{Q}[\sqrt{2}.\sqrt{3}.\sqrt{5}.\sqrt{7}] - \mathbb{Q}[\sqrt{2}.\sqrt{3}.\sqrt{5}] \\ 0 & x \in \mathbb{Q}[\sqrt{2}.\sqrt{3}.\sqrt{5}.\sqrt{7}.\dots] = \mathbb{R} \end{cases}$$

Since for each $t \in [0.1]$, A_t is a *G-domain* as the follows:
$A_0 = \mathbb{Q}[\sqrt{2}.\sqrt{3}.\sqrt{5}.\sqrt{7}.\dots] = \mathbb{R}\,.A_{1/7} = \mathbb{Q}[\sqrt{2}.\sqrt{3}.\sqrt{5}.\sqrt{7}]$. $A_{1/5} = \mathbb{Q}[\sqrt{2}.\sqrt{3}.\sqrt{5}]$. $A_{1/3} = \mathbb{Q}[\sqrt{2}.\sqrt{3}]$. $A_{1/2} = \mathbb{Q}[\sqrt{2}]$ *and* $A_1 = \mathbb{Q}$. Hence A is a *fuzzyG-subdomain* of $\mathbb{R}$.

Lemma 2.2. If *A* be a *fuzzy G-subdomain* of *D* with the quotient fuzzy subfield *K* for D_A, then For $0 \neq u \in D$ we have:

$$K = D_A[u^{-1}]$$

Lemma 2.3. *Let A be a fuzzy subdomain of D and u be an element located on a domain containing* D_A . *if* $D_A[u]$ *is a G-subdomain, then: A is a fuzzy G-subdomain of D and u is algebraic on* D_A .

Theorem 2.1. *A is a fuzzy G-subdomain of D if and only if for each* $t \in [0.1]$. A_t *is a G-subdomain of* D_A .

Proof. Let A be a fuzzy *G-subdomain* of *D* and $t \in [0.1]$, since $A(0) \geq A(x). \forall x \in D_A$, then: $0 \in A_t. therefore\ A_t \neq \emptyset$ .Now let $x.y \in A_t. since\ A(x) \geq t\ .\ A(y) \geq t\ and\ A$ is a fuzzy subdomain of *D*, then:

$$A(xy) \geq inf\{A(x)\,.A(y)\} = A(x) \wedge A(y) \geq t \wedge t = t$$
$$\Rightarrow A(xy) \geq t \Rightarrow xy \in A_t$$

Furthermore:

$$A(x-y) \geq \inf\{\,A(x)\,.A(y)\} = A(x) \wedge A(y) \geq t \wedge t = t$$
$$\Rightarrow A(x - y) \geq t \Rightarrow x - y \in A_t$$

Therefore A_t is a fuzzy subdomain of D_A and so it's a fuzzy subdomain of *D*.
In addition, for the quotient field of *K* related to D_A, we have $K = D_A[u^{-1}]$. $\exists\, 0 \neq u \in K$ and since A_t is a subdomain of D_A, so

$$A_t \leq D_A \Rightarrow A_t[u^{-1}] \leq D_A[u^{-1}] = K$$

Since by Lemma "2.3" u is algebraic on A_t, then A_t is a G-subdomain of D_A.
Conversely, let A_t be a G - subdomain of D_A, $\forall t \in A(D_A)$.Since $0 \in A_t$, then $A(0) \geq t$.

Let $x.y \in D_A$ and let $A(x) = t_1.\ A(y) = t_2\ and\ t_3 = t_1 \wedge t_2\ so\ x.y \in D_{A_{t_3}}$, therefore $t_3 \leq A(0)$ .therefore $D_{A_{t_3}}$ is a subdomain of *D* and hence $xy \in D_{A_{t_3}}$.

Now:

$$A(xy) \geq t_3 = t_1 \wedge t_2 = A(x) \wedge A(y)$$

Since *A* has the G- structure property, therefore *A* is a fuzzy *G*-subdomain of D_A and hence *A* is a fuzzy *G*-subdomain of *D*. □

Theorem 2.2. *Let A is a fuzzy G-subdomain of D with the quotient field K of D_A and let B be a fuzzy subring, such that:*

$$D_A \subseteq D_B \subseteq K$$

Then B is a fuzzy G-subdomain of D.

Proof. Since $K = D_A[u^{-1}]$ for some $u \in K$, then $K = D_A[u^{-1}] = D_B[u^{-1}]$, therefore *B* is a fuzzy *G*-subdomain of *D*. □

Theorem 2.3. *Let $A.B \in F(D)$ be arbitrary members, if $A \subset B$ and D_B is algebraic on D_A and D_B as a subring above D_A is finitely generated, then:*
A is a fuzzy G-subdomain of D if and only if B is a fuzzy G-subdomain of D.

Proof. Let K , L be the quotient fuzzy subfield of $D_A . D_B$. Suppose first that A is a fuzzy G-subdomain, say $K = D_A[u^{-1}]$. Then $D_B[u^{-1}]$ is a fuzzy subdomain algebraic over the quotient field of K, hence itself is a field, necessarily equal to L.Thus B is an fuzzyG-subdomain of D.
Conversely,assume that B is a fuzzy G-subdomain of D, $L = D_B[v^{-1}]$ and $D_B = D_A[w_1.w_2.\cdots.w_k]$. The elements $v^{-1}.w_1.w_2.\cdots.w_k$ are algebraic over D_A and consequently satisfy equations with coefficients in D_A which lead off, say:

$$av^{-m} + \cdots = 0$$

$$b_i w_i^{n_i} + \cdots = 0 \qquad (i = 1.\cdots.k)$$

Adjoin $a^{-1}.b_1^{-1}.\cdots.b_k^{-1}$ to D_A, obtaining a subring D_1 between D_A and K. The field L is generated over D_A by $w_1.\cdots.w_k.v^{-1}$. Of course these elements generate L over D_1. Now over D_1 we have arranged that $w_1.\cdots.w_k.v^{-1}$ are integral. Hence L is integral over D_1.Therefore, D_1 is a field, necessarily K. So K is a finitely generated ring over $D_A$and therefore A is afuzzy G-subdomain of D, as required. □

Theorem 2.4. *Let A be a fuzzy subdomain of D and u is an element of a larger of D_A , if $D_A[u]$ is a G-domain, then u is algebraic over D_A and A is a fuzzy G-subdomain of D.*

Proof. Let $T_A = D_A[u]$ is a G-domain if we define $L_A = T_A[v^{-1}]$, then $u.v^{-1}$ are algebraic on D_A, because for each $\alpha \in T_A$, $t \in D_A[u]$, in especial case we have $0 \in T_A$, then:

$$0 = a_0 + a_1 u + \cdots + a_n u^n$$

So u is algebraic on D_A .
On the other hand T_A as a ring on D_A is finitely generated, therefore by Theorem "2.3" D_A is G-domain and hence A is fuzzy G-subdomain of D. □

Theorem 2.5. *Let A and B are two fuzzy subdomains of D such that $D_A \subseteq D_B$ and D_B finitely generated as a ring over D_A .and D_B is not algebraic over D_A, then B is not a fuzzy G-subdomain of D.*

Proof. Let $D_B = D_A[\alpha_1.\alpha_2.\cdots.\alpha_n]$and let $\alpha_{m+1}.\cdots.\alpha_n$ are not algebraic over D_A. By Theorem"2.4" $D_A[\alpha_{m+1}]$ is not a *G*-domain.
Put $D_{A_0} = D_A[\alpha_1.\alpha_2.\cdots.\alpha_m][\alpha_{m+1}]$ *.then* D_{A_0} is not a G-domain.Again, consider α_{m+2}. if α_{m+2} is not algebraic over D_{A_0} then $D_{A_1} = D_{A_0}[\alpha_{m+2}]$ is never an G-domain and if α_{m+2} is not algebraic over D_{A_0}.then D_{A_1} is not a *G*-domain. However D_{A_1} is not a G-domain. By repetition this process we obtain that $D_B$is not a *G*-domain and therefore B is not a fuzzy *G*-subdomain of *D*. □

Lemma 2.4.*Let R be a ring and A is a fuzzy subring of R,*
1) If M is an ideal of $R_A[x]$ *and satisfying in* $M \cap R_A = 0$ *. Let ube the image of x under the canonical homomorphism* $R_A[x] \longrightarrow \frac{R_A[x]}{M}$ *.then:*

$$\frac{R_A[x]}{M} \simeq R_A[u]$$

In addition, if M is maximal in $R_A[x]$ *, then the* $R_A[u]$ *is a field.*
2) In general form, if M is a maximal ideal of $\mathcal{R}_A = R_A[x_1.x_2.\cdots.x_n.\cdots]$ *satisfy* $M \cap R_A = 0$*, and if* u_i *is the image of* $x_i + M$ *for each* $i \in \aleph_0$ *under the canonical homomorphism*

$$\frac{\mathcal{R}_A}{M} \longrightarrow \mathcal{R}_A \quad . \quad \forall\, i \in \aleph_0$$

Therefore

$$\frac{\mathcal{R}_A}{M} \simeq R_A[u_1.u_2.\cdots.u_n.\cdots]$$

Proof.1) At first, the note $R_A \cap M = 0$ requires that $R_A \simeq \{a + M: a \in R_A\}$.which is a subring of $R[x]/M$ (say: S_A). Now,

$$R_A[u] \simeq S_A[x+M] = \{\sum_{i=0}^{n} s_i(x+M)^i \,:\, s_i \in S_A.n \in \aleph_0\}$$

$$= \{\sum_{i=0}^{n}(a_i + M)(x+M)^i \,:\, a_i \in R_A.n \in \aleph_0\}$$

$$= \{\sum_{i=0}^{n}(a_i + M)\big(x^i + M\big) : a_i \in R_A.n \in \aleph_0\}$$

$$= \{\sum_{i=0}^{n}\big(a_i x^i + M\big) : a_i \in R_A.n \in \aleph_0\}$$

$$= \left\{\sum_{i=0}^{n}(f(x) + M) : f(x) \in R_A[x]\right\} = \frac{R_A[x]}{M}$$

2) By similar argument that is expressed in part "1" and this fact that any element of $\mathcal{R}_A$ is of the form:

$$F(\bar{X}) = \{\sum_{(i_1.i_2.\cdots.i_n)\in\Lambda} a_{i_1.i_2.\ldots.i_n} x_1^{i_1}\cdots x_n^{i_n} \quad .for\ some\ finite\ subset\ \Lambda \subseteq \aleph_0^n\}$$

We have:

$$R_A[u_1.u_2.\cdots.u_n.\cdots] \simeq S_A[x_1+M.x_2+M.\cdots.x_n+M.\cdots]$$

$$= \{\textstyle\sum_{(i_1.i_2.\cdots.i_n)\in\Lambda} s_{i_1.i_2.\ldots.i_n}(x_1+M)^{i_1}\ldots(x_n+M)^{i_n} \ .s_{i_1.i_2.\ldots.i_n} \in S_A\}$$

$$= \{\textstyle\sum_{(i_1.i_2.\cdots.i_n)\in\Lambda}(a_{i_1.i_2.\ldots.i_n}+M)(x_1+M)^{i_1}\ldots(x_n+M)^{i_n} \ .a_{i_1.i_2.\ldots.i_n} \in R_A\}$$

$$= \{\textstyle\sum_{(i_1.i_2.\cdots.i_n)\in\Lambda}(a_{i_1.i_2.\ldots.i_n}+M)\left(x_1^{i_1}+M\right)\ \ldots\left(x_n^{i_n}+M\right)\ .a_{i_1.i_2.\ldots.i_n} \in R_A\}$$

$$= \left\{\sum_{(i_1.i_2.\ldots.i_n)\in\Lambda} a_{i_1.i_2.\ldots.i_n} x_1^{i_1}\ldots x_n^{i_n} \quad .for\ some\ finite\ subset\ \Lambda \subseteq \aleph_0^n\ .a_{i_1.i_2.\ldots.i_n} \in R_A\right\}$$

$$= \{F(\bar{x})+M.\ \ where\ F(\bar{x}) is\ a\ typical\ element\ of\ \mathcal{R}_A\} = \frac{\mathcal{R}_A}{M}$$

Which is proved the second part. □

Theorem 2.6.*The fuzzy subset A of an integral domain D is a fuzzy G-subdomain of D if and only if there exists in the polynomial ring $D_A[x]$ an ideal M which is maximal and satisfies $M \cap D_A = 0$.*

Proof. Let A be a fuzzy G-subdomain of D, we have: $K = D_A(u^{-1})$, where K is a quotient field of D_A. Now we define the ring homomorphism as the following:

$$\begin{aligned} \phi: \mathrm{D_A[x]} &\longrightarrow \mathrm{K} = \mathrm{D_A[u^{-1}]} \\ \mathrm{x} &\longmapsto \mathrm{u^{-1}} \end{aligned}$$

it is obvious that the image is all of K, so the kernel M is maximal.
Since the homomorphism is one to one on R, therefore we have $M \cap D_A = 0$.
Conversely, let M be maximal ideal in $D_A[x]$ and satisfying in $M \cap D_A = 0$. Denote the image of x in the natural homomorphism $D_A[x] \longrightarrow \frac{D_A[x]}{M}$ by ν . Then by lemma"2.3" $D_A[\nu]$ is a field and therefore by Theorem "2.4" D_A is a *G*-domain and hence *A* is a fuzzy *G*-subdomain of *D*.

It is suggested that further research in this direction is likely going to reveal additional properties of fuzzy G-ideals associated to fuzzy subdomains and thus contribute to our understanding of how such structures defines on the underlying fuzzy G-subdomains.

Acknowledgment.
The author would like to thank the referee for the valuable suggestions and comments.

References

[1] A.R. Alizadeh M., *On Fuzzy G-ideals*, IJISET, Vol. 3, Issue. 10, Pp. 51-56 (2016).
[2]A.R. Alizadeh M., *On Fuzzy G-domains*, IJISET, Vol. 3, Issue. 7, Pp. 328-334 (2016).
[3]V. N. Dixit, R. Kumar and N. Ajmal, *On Fuzzy Rings*, Fuzzy Sets and Systems, Vol. 49, Pp. 205-213 (1992).
[4]R.Gilmer, *The cancellation law for ideals in a Commutative Ring*, Canad. J. Math. Vol. 17, Pp. 281-287 (1965).
[5]O.Goldman, *Hilbert Ring and Hilbert Nullstellensatz*, Math. Z. Vol. 54, Pp. 136-140 (1951).
[6]I. Kaplansky, *Commutative Rings*, Allyn and Bacon,Boston, (1974).
[7]W.Krull, *Jacobsonsche, Hilbertsche, Nullstellensatz*, Math. Z. Vol. 54, Pp. 354-387 (1951).
[8] H.V. Kumbhojkar and M.S. Bapat, *On Prime and Primary Fuzzy Ideals and Their Radicals*, Fuzzy Sets and Systems, Vol. 37, Pp. 237-243 (1990).
[9]M. Mashinchi, M.M. Zahedi, *On fuzzy ideals of a ring*, J. Sci. I. R. Iran, Vol. 1, Issue. 3, Pp. 208-210 (1990).
[10]J.N. Mordeson, and D.S. Malek, *Fuzzy Commutative Algebra*, World Scientific Publishing, Singapore, (1999).
[11] K.L.N. Swamy and U.M. Swamy, *Fuzzy Prime Ideals of Ring*, J. Math. Anal. Appl., Vol. 13, Pp. 94-103, (1988).
[12] L.A. Zadeh, *Fuzzy Sets*, Inform. Control, Vol. 8, Pp. 338-353, (1965).