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Triangular Difference Graphs

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Abstract
A (p, q) graph is said to admit triangular difference labeling if its vertex can be labeled by positive integers such that the induced edge labels obtained by the absolute difference of the labels of end vertices are the triangular numbers. A graph G which admit triangular difference labeling is called Triangular difference graph. In this paper, we prove path, combs, stars, bistars and subdivision of stars admit Triangular Difference labeling and also cycle C_n *is not Triangular Difference graph when* $n \equiv 1(mod\ 4)$ *and complete graph* k_n *doesn't admit triangular difference labeling. Also* C_n *admits Triangular Difference labeling iff the set of all first* n *triangular numbers is a zero sum set.*
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Keywords
Triangular Difference graph, zero sum set

Introduction

All graphs in this paper are finite, simple and undirected. The vertex set and edge set of graph G are denoted by V(G) and E(G) respectively. For various graph theoretic notations and terminology we follow Harry [1] and for number theory we follow Burton [2]. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. A dynamic survey on graph labeling is regularly updated by Gallian [3] and it is published by Electronic Journal of Combinatorics. Vast amount of literature is available on different types of graph labeling and more than 1000 research papers have been published so far in last four decades.

Definition 1.1

A triangular number is a number obtained by adding all positive integers less than or equal to a given positive integer n . If the n^{th} triangular number is denoted by A_n then

$$A_n = 1 + 2 + \ldots + n$$

The triangular numbers are 1,3,6,10,15,21,28,36,45,55,66,78,...

Definition 1.2
A triangular difference labeling of a graph G is a one-one function $f: V(G) \rightarrow \mathrm{N}$ (where N is set of non-negative integers) that induces a bijection $f^{+}: E(G) \rightarrow \{A_1, A_2, \ldots, A_q\}$ of the edges defined by $f^{+}(uv) = |f(u) - f(v)|\ \forall\ e = uv \in$ E(G). The graph which admit such labeling is called a Triangular Difference graph.

Definition 1.3
A finite subset F_n of positive integers is said to be zero sum set if there exist a linear combination, $\sum_{i=1}^{n} a_i x_i = 0$, $x_i \in F_n$ for $i = 1,2\ldots, n$ and $a_i \in \{-1,1\}$ for $i = 1,2, \ldots, n$.

Definition 1.4
A subset M with k elements of a finite set of positive integers F_n is said to be k −zero sum subset, if M itself a zero sum set.

Definition 1.5
The n^{th} sum of triangular numbers is denoted by S_{A_n} ,

$$S_{A_n} = \sum_{i=1}^{n} A_i$$

Main Results

Here we prove that path, combs, stars, bistars, coconut trees and subdivision of stars admit Triangular Difference labeling. The complete graph K_n is not triangular difference graph and cycle C_n is not Triangular Difference graph when $n \equiv 1 (mod\ 4)$. Also C_n admits Triangular Difference labeling iff the set of all first n triangular numbers is a zero sum set.

Theorem 2.1
The path P_n admits triangular difference labeling.
Proof:
Let $P_n : u_1, u_2, \ldots\ldots, u_n$ be the path and let $w_i = u_i u_{i+1}$ (1≤ $i \leq n$) be the edges.
for $i = 1,2, \ldots, n$, defined

$$f(u_i) = \frac{(\mathrm{i}-1)\mathrm{i}(\mathrm{i}+1)}{6}$$

We will show that the induced edge labels obtained by the absolute difference of end vertices are the first $(n-1)$ triangular numbers.

For $1 \leq i \leq n-1$,

$$|f(u_i) - f(u_{i+1})| = \left|\frac{(\mathrm{i}-1)\mathrm{i}(\mathrm{i}+1)}{6} - \frac{i(\mathrm{i}+1)(\mathrm{i}+2)}{6}\right|$$
$$= \left|\frac{\mathrm{i}(\mathrm{i}+1)[\mathrm{i}-1-(\mathrm{i}+2)]}{6}\right|$$

$$= \frac{i(i+1)}{2}$$
$$= A_i$$
$$= f^i(w_i)$$

Thus the induced edge labels are the first $n-1$ triangular numbers.
Thus path P_n admits triangular difference labeling .

Example 2.2
The triangular difference labeling of P_6 is shown below.

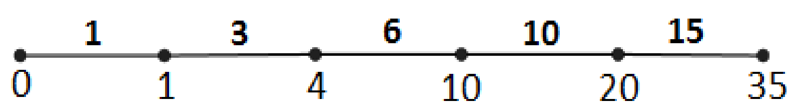

Figure 1

Theorem 2.3
The comb $P_n \odot K_1$ admits triangular difference labeling .
Proof:
Let $P_n: u_1, u_2, \ldots\ldots, u_n$ be the path and let $w_i = u_i u_{i+1}$ $(1 \le i \le n-1)$ be the edges. Let $v_1, v_2, \ldots\ldots, v_n$ be the pendant vertices adjacent to $u_1, u_2, \ldots\ldots, u_n$ respectively and $t_i = u_i w_i$ $(1 \le i \le n)$ be the edges.

for $i = 1,2,\ldots,n,$ define
$$f(u_i) = \frac{(i-1)i(i+1)}{6}$$
$$f(v_i) = \frac{i^3 - 4i + 3(n^2 + 2ni + i^2 - n)}{6}$$

Then,
$$|f(u_i) - f(u_{i+1})| = f^+(w_i) \quad for\ 1 \le i \le n-1$$
$$|f(u_i) - f(v\_i)| = f^i(t_i) \quad for\ 1 \le i \le n$$

Thus the induced edge labels are the first $(2n-1)$ triangular numbers.
Hence comb $P_n \odot K_1$ admits triangular difference labeling .
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Example 2.4

The 2-Product labeling of $P_5 \odot K_1$ is shown below.

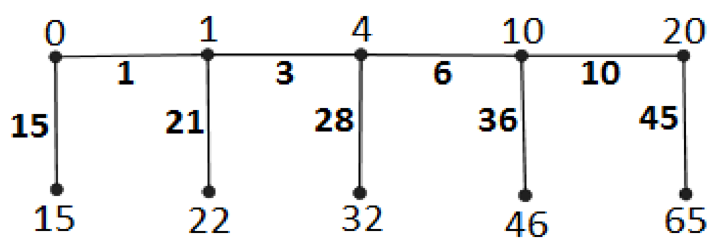

Figure 2

Theorem 2.5

The star graph $K_{1,n}$ admits triangular difference labeling .
t labeling.

Proof:

Let u be the apex vertex and let $v_1, v_2, \ldots\ldots, v_n$ be the pendant vertices of the star $K_{1,n}$.
Define,

$$f(u) = 1$$
$$f(u_i) = \frac{i(i+1)}{2} + 1 \quad for\ 1 \le i \le n-1$$

We see that induced edge labels are the first n triangular numbers.
Hence star graph $K_{1,n}$ admits triangular difference labeling.

Example 2.6

The triangular difference labeling of $K_{1,6}$ is shown below.

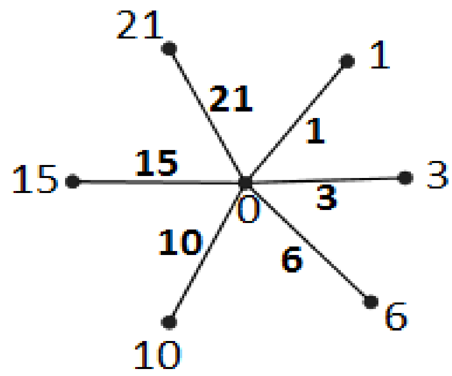

Figure 3

Theorem 2.7
The bistar $B_{m,n}$ admits triangular difference labeling.

Proof:
Let, $V(B_{m,n}) = \{u, v, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$,
$E(B_{m,n}) = \{uv, uu_i, vv_j : 1 \leq i \leq m, 1 \leq j \leq n\}$
Define f by,

$$f(u) = 0$$
$$f(v) = 1$$
$$f(u_i) = \frac{(i+1)(i+2)}{2} \quad \text{for } 1 \leq i \leq m$$
$$f(v_j) = \frac{(m+2+i)(m+3+i)}{2} + 1 \quad for\ 1 \leq j \leq n$$

we see that induced edge labels are the first $m + n + 1$ triangular numbers.
Hence bistar $B_{m,n}$ admits triangular difference labeling.

Example: 2.8
The triangular difference labeling of $B_{4,3}$ is shown below.

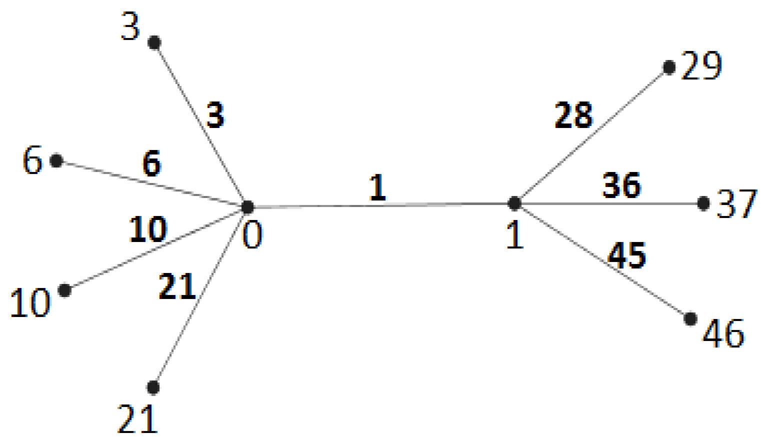

Figure: 4

Theorem: 2.9
$S(K_{1,n})$ the subdivision of the star $K_{1,n}$,$admits triangular difference labeling.
Proof:
Let $S(K_{1,n})$ ={v, v_i, u_i: $1 \leq i \leq n$} .
Define f by,

$$f(v) = 0$$

$$f(v_i) = \frac{i(i+1)}{2} \quad for\ 1 \leq i \leq n$$

$$\mathrm{f(u_i)} = \frac{\mathrm{i(i+1) + (n+i)(n+i+1)}}{2} \quad \mathrm{for}\ 1 \leq \mathrm{i} \leq \mathrm{n}$$

we see that induced edge labels are the first $m+n-1$ triangular numbers.
Hence, $S(K_{1,n})$ the subdivision of the star $K_{1,n}$ admits triangular difference labeling.

Example: 2.10
The triangular difference labeling of $S(K_{1,n})$ is shown below.

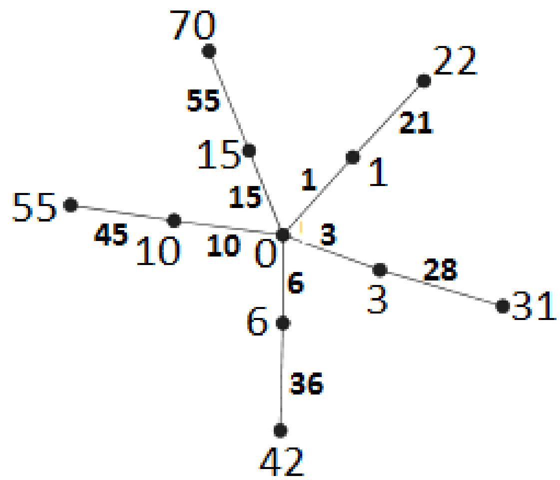

Figure: 5

Result: 2.11
The n^{th} sum of triangular numbers is denoted by S_{A_n},

$$S_{A_n} = \frac{n(n+1)(n+2)}{6}.$$

Theorem: 2.12 [3]
If F_n be a zero sum set then S_{F_n} is an even number, where $S_{F_n} = \sum_{i=1}^{n} x_i$, $where\ x_i \in F_n$.

Theorem: 2.13
An n −cycle C_n admits triangular difference labeling iff $\{A_1, A_2, \ldots\ldots, A_n\}$ is a zero sum set.
Proof:
Suppose C_n admits triangular difference labeling.
Let $C_n : u_1, u_2, \ldots\ldots, u_n, u_1$ be the n −cycle.
Let $f(u_i) = x_i$ for $i = 1,2,\ldots,n$.
Then, we have,

$$|x_1 - x_2| = A_{i_1}$$
$$|x_2 - x_3| = A_{i_2}$$
$$\vdots$$
$$|x_n - x_1| = A_{i_n}$$

Where, $A_{i_j} \in \{A_1, A_2, \ldots\ldots, A_n\}$
Above equations can be written as

$$x_1 - x_2 = a_{i_1} A_{i_1} \qquad (1)$$
$$x_2 - x_3 = a_{i_2} A_{i_2} \qquad (2)$$
$$\vdots$$
$$x_n - x_1 = a_{i_n} A_{i_n} \qquad (n)$$

Adding $(1) + (2) + \ldots .. + (n)$, we get

$$\sum_{j=1}^{n} a_{i_j} A_{i_j} = 0$$

Which implies, $\sum_{i=1}^{n} a_i A_i = 0$

Thus $\{A_1, A_2, \ldots\ldots, A_n\}$ is a zero sum set.
Conversly, suppose $\{A_1, A_2, \ldots\ldots, A_n\}$ is azero sum set. We have to prove n-cycle C_n admits triangular difference labeling.
Since, $\sum_{i=1}^{n} a_i A_i = 0$ for $a_i \in \{1, -1\}$
Thus , we have,

$$A_{i_1} + A_{i_2} + \cdots + A_{i_k} - A_{i_{k+1}} - A_{i_{(k+1)}} - \cdots . - A_{i_n} = 0$$

Where , $A_{i_j} \in \{A_1, A_2, \ldots\ldots, A_n\}$
Now, we define ,

$$f(u_1) = 1$$
$$f(u_{j+1}) = f(u_1) + A_{i_j} \quad for\ \ j = 1,2, \ldots, k.$$
$$f(u_{j+1}) = f(u_j) - A_{i_j} \quad for\ \ j = k+1, \ldots, n.$$

we see that induced edge labels are the first n triangular numbers.
Thus C_n is triangular difference graph.

Corollary 2.14
Let (p, q) graph G be a Triangular Difference graph with k-cycle , Then there is a k-zero sum subset in $\{A_n : n = 1,2. \ldots, q\}$.

Theorem: 2.15
The complete graph K_n doesn't admit triangular difference labeling.
Proof:
In K_n for any three vertices x_1, x_2, x_3 , there is a 3-cycle connecting vertices x_1, x_2, x_3.
Since $1 \in E(G)$, then , there is $3-$cycle with edges $1, A_{i_1}, A_{i_2}$ where $A_{i_1}, A_{i_2} \in \{A_i : n = 1,2 \ldots, n\}$.
By **Corollary 2.14,**
$\{1, A_{i_1}, A_{i_2}\}$ is a zero sum subset . But it is impossible.
Thus, complete graph K_n doesn't admit triangular difference labeling.

Theorem 2.16
A cycle C_n doesn't admits triangular difference labeling for $n \equiv 1 \ (mod\ 4)$.

Proof:

For $n = 4k + 1$, for integer k, The n^{th} sum of triangular numbers is $S_{A_n} = \frac{n(n+1)(n+2)}{6}$ is an odd number.

For, we have to show $S_{A_{4k+1}} = \frac{(4k+1)(4k+2)(4k+3)}{6}$ is not congruent to 0(mod 2).

For that, $(4k + 1)(4k + 2)(4k + 3) \equiv 64k^3 + 6(mod\ 12)$

By Division algorithm, consider the cases for $k = 3s, k = 3s + 1$ and $k = 3s + 2$ for integer s.

For k=3s,

$$64(3s)^3 + 6 \equiv 192s^3 + 6\ (mod\ 12)$$
$$\equiv 6\ (mod\ 12)$$

Thus , $64(3s)^3 + 6$ is not congruent to 0 (mod 12) .

Similarly, for $k = 3s + 1$, and $k = 3s + 2$, It can be easily verify that $64(3s)^3 + 6$ is not congruent to 0 (mod 12) .

Thus $S_{A_{4k+1}}$ is an odd number .

Therefore , $\{A_1, A_2, \ldots\ldots, A_{(4k+1)}\}$ is not a zero sum set .

Thus by **theorem: 2.13 ,** Cycle C_n is not triangular difference graph for $n \equiv 1\ (mod\ 4)$.

Concluding Remarks

In this paper, we introduce a new concept of Triangular difference graph labeling, we proved Triangular difference graph labeling of standard graphs and introduced zero sum set and its application in labeling of graphs. To investigate analogues result for different graphs and extend the labeling further towards a generalization is an open area of research.

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