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Solution for n^{th} order mixed Fredholm-Volterraintegro-differential equations using Haar wavelets

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Abstract:
In this article a collocation method based on Haar wavelets is presented for the solution of n^{th} order linear mixed Fredholm-Volterraintegro-differential equations. The solution process is illustrated by designing general form of matrices to make calculations in easy way. Numerical experiments are done on test problems to examine the capability and efficiency of proposed method. Numerical and graphical results with comparison are given to demonstrate the accuracy and reasonability of this technique.

Keywords: *Collocation method, Collocation points, Haar wavelets, Integro-differential equations*
MSC 2010 No: 65T60,45J05, 65L60.

1. Introduction

Wavelets are non-periodic and have compact support unlike the Fourier basis. They form basis for $L^2(R)$ through process of scaling and translation of mother wavelet. They combine powerful properties which are orthogonally, compact support, localization in time and frequency and fast algorithms. Wavelets have many applications in data compression, noise removal, nuclear engineering, signal and image processing, radar, telecommunications etc. Wavelets are in many types viz. Haar, Daubechies, Bi-orthogonal, Coiflets, Morlet, Mexican hat and Meyer etc [1]. Alfred Haar [2] introduced the idea of wavelets and these wavelets became important tool for the solution of differential and integral equations. There are two methods to applying Haar wavelets for integrating ordinary differential equations. In first method Chen and Hsiao [3, 4] introduced the concept of operational matrix for integrating ODE. In second method Lepik integrated Haar functions directly and called as direct method [5]Integro-differential equations (IDE) arise in the mathematical modeling of real-life problems. These equations have many applications in science and engineering. Some fields are glass forming process, nanohydrodynamics, theoretical development of boundary value problems, biological and physical models, chemical kinetics, fluid dynamics etc [6].The mixed Fredholm-Volterraintegro-differential equation is the combination of Fredholmintegro-differential equation and Volterraintegro-differential equation. Many researchers have been studied the solutions of mixed integro-differential equations using different numerical methods. Yuzbasi et al. [7] have used the Bessel polynomial functions, Rahmani et al. [8] have applied block pulse functions and operational matrices, Gulsu et al. [9] have employed Chebyshev collocation method and Ordokhani et al. [10] have applied Bessel hybrid functions. Haar wavelets are used for the solution of integro-differential equations by Mishra et al. [11], Siraj-ul-islam et al. [12] and Lepik [13].

In this paper Haar wavelet collocation method has been used for solving n^{th} order linear mixed Fredholm-Volterraintegro-differential equations of second kind over $[a,b]$ is of following form:

$$y^{(n)}(x) = g(x) + \int_a^b k_1(x,t) y^{(m1)}(t) dt + \int_a^x k_2(x,t) y^{(m2)}(t) dt, \quad (1.1)$$

subject to: $\beta_i = y^{(n-i)}(a), \quad i = 1,2,...,n \quad$ where $n \geq \max\{m_1, m_2\}$,

$k_1(x,t), k_2(x,t), g(x)$ are known functions and $y^{(n)}(x),\ y^{(m_1)}(t),\ y^{(m_2)}(t)$ are the derivatives of unknown function y.

This article is organized as in section 2, Haar wavelets are introduced. In section 3, algorithm based on Haar wavelets and design of matrices is presented. The algorithm is examined on few problems in section 4. The conclusion has been inserted in final section.

2. Haar Wavelets

We considered the interval $t \in [a,b]$. Where a and b are given constants. The interval $[a,b]$ is divided into 2^{J+1} subintervals of equal length. The length of each subinterval is $\Delta t = \frac{(b-a)}{2^{J+1}}$. Here J indicates maximal level of resolution. Another two parameters are dilation: $j = 0,1,2,...,J$ and translation: $k = 0,1,2,...,2^j - 1$ [1, 14, 15]. By these parameters i^{th} Haar wavelet is defined as

$$h_i(t) = \begin{cases} 1, & for\ t \in [\tau_1(i), \tau_2(i)), \\ -1, & for\ t \in [\tau_2(i), \tau_3(i)), \\ 0, & otherwise, \end{cases} \quad (2.1)$$

here $i = m + k + 1,\ \tau_1(i) = a + 2k\mu\Delta t,\ \tau_2(i) = a + (2k+1)\mu\Delta t$ and $\tau_3(i) = a + 2(k+1)\mu\Delta t$, where $\mu = 2^{J-j}$. (2.1) is valid for $2 < i \leq 2^{J+1}$.

For $i = 1$ we have

$$h_1(t) = \begin{cases} 1, & for\ t \in [a,b), \\ 0, & otherwise, \end{cases} \quad (2.2)$$

h_1 is called a father wavelet. For $i = 2$ we have

$$h_2(t) = \begin{cases} 1, & for\ t \in \left[a, \frac{a+b}{2}\right), \\ -1, & for\ t \in \left[\frac{a+b}{2}, b\right), \\ 0, & otherwise, \end{cases} \quad (2.3)$$

h_2 is called a mother wavelet. Any function which is having finite energy on $[a,b]$ i.e. $f \in L^2[a,b]$ can be decomposed as infinite sum of Haar wavelets:

$$y(x)=\sum_{i=1}^{\infty}a_i h_i(x). \tag{2.4}$$

Here, a_i's are Haar coefficients. If f is either piecewise constant or wish to approximate by piecewise constant on each subinterval, then the above infinite series will be terminated at a finite number of terms.

3. Haar wavelet collocation method for n^{th} order mixed IDE

To solve above defined mixed n^{th} order IDE (1.1), unknown function with highest derivative defined on $x\in[0,1]$ is approximated by truncated Haar wavelets:

$$y^{(n)}(x)=\sum_{i=1}^{2^{J+1}}a_i h_i(t). \tag{3.1}$$

Integrate equation (3.1) n times, yields

$$y(x)=\sum_{i=1}^{2^{J+1}}a_i\int_0^x\ldots\int_0^x h_i(t)dt+\sum_{\nu=0}^{n-1}\frac{x^\nu}{\nu!}\beta_{n-\nu} \tag{3.2}$$

$$\int_0^x\ldots\int_0^x h_i(t)dt=\frac{1}{(n-1)!}\int_0^x(x-t)^{n-1}h_i(t)dt,$$

Equation (3.2) implies,

$$y(x)=\sum_{i=1}^{2^{J+1}}a_i\frac{1}{(n-1)!}\int_0^x(x-t)^{n-1}h_i(t)dt+\sum_{\nu=0}^{n-1}\frac{x^\nu}{\nu!}\beta_{n-\nu} \tag{3.3}$$

Let $K_n(x,t)=\dfrac{(x-t)^{n-1}}{(n-1)!}$, $P_{n,i}(x)=\int_0^x K_n(x,t)h_i(t)dt,$ where $i=1,2,\ldots,2^{J+1}.$ (3.4)

$$y(x)=\sum_{i=1}^{2^{J+1}}a_i P_{n,i}(x)+\sum_{\nu=0}^{n-1}\frac{x^\nu}{\nu!}\beta_{n-\nu}. \tag{3.5}$$

Similarly,

$$y^{(m_1)}(x)=\sum_{i=1}^{2^{J+1}}a_i P_{n-m_1,i}(x)+\sum_{\nu=0}^{n-m_1-1}\frac{x^\nu}{\nu!}\beta_{n-m_1-\nu}, \qquad 1\le m_1\le n, \tag{3.6}$$

$$y^{(m_2)}(x)=\sum_{i=1}^{2^{J+1}}a_i P_{n-m_2,i}(x)+\sum_{\nu=0}^{n-m_2-1}\frac{x^\nu}{\nu!}\beta_{n-m_2-\nu}, \qquad 1\le m_2\le n. \tag{3.7}$$

Substituting equations (3.1), (3.6) and (3.7) into (1.1) we get

$$\sum_{i=1}^{2^{J+1}} a_i h_i(x) = g(x) + \int_0^1 k_1(x,t)\left\{\sum_{i=1}^{2^{J+1}} a_i P_{n-m_1,i}(t) + \sum_{\nu=0}^{n-m_1-1} \frac{x^\nu}{\nu!}\beta_{n-m_1-\nu}\right\}$$

$$+\int_0^x k_2(x,t)\left\{\sum_{i=1}^{2^{J+1}} a_i P_{n-m_2,i}(t) + \sum_{\nu=0}^{n-m_2-1} \frac{x^\nu}{\nu!}\beta_{n-m_2-\nu}\right\},$$

or,

$$\sum_{i=1}^{2^{J+1}} a_i\left[h_i(x) - E_i(x) - G_i(x)\right] = g(x) + \sum_{\nu=0}^{n-m_1-1}\frac{\beta_{n-m_1-\nu}}{\nu!}T_\nu(x) + \sum_{\nu=0}^{n-m_2-1}\frac{\beta_{n-m_2-\nu}}{\nu!}U_\nu(x), \qquad (3.8)$$

here $E_i(x) = \int_0^1 k_1(x,t)P_{n-m_1,i}(t)dt$, $T_\nu(x) = \int_0^1 k_1(x,t)t^\nu dt$ where $i = 1,2,...,2^{J+1}$, $\nu = 0,1,...,n-m_1-1.$

$G_i(x) = \int_0^x k_2(x,t)P_{n-m_2,i}(t)dt$, $U_\nu(x) = \int_0^x k_2(x,t)t^\nu dt$, $\nu = 0,1,...,n-m_2-1.$

Equation (3.4) implies that if $i=1$, then

$$P_{n,1}(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!}h_1(t)dt = \frac{x^n}{n!} \ , \qquad (3.9)$$

if $i \ge 2$, we have

$$P_{n,i}(x) = \begin{cases} 0, & if \ x \in [0,\tau_1(i)), \\ \frac{1}{n!}(x-\tau_1(i))^n, & if \ x\in[\tau_1(i),\tau_2(i)), \\ \frac{1}{n!}\left\{(x-\tau_1(i))^n - 2(x-\tau_2(i))^n\right\}, & if \ x\in[\tau_2(i),\tau_3(i)), \\ \frac{1}{n!}\left\{(x-\tau_1(i))^n - 2(x-\tau_2(i))^n + (x-\tau_3(i))^n\right\}, & if \ x\in[\tau_3(i),1). \end{cases} \qquad (3.10)$$

The matrix E with $(i,l)^{th}$ entry is constructed as

$$E_{i,l} = E_i(x_l) = \frac{1}{(n-m_1)!}\begin{cases} \int_0^1 k_1(x_l,t)t^{n-m_1}dt, & i=1, \\ \int_0^1 k_1(x_l,t)(t-\tau_1)^{n-m_1}dt + \int_0^1 k_2(x_l,t)(t-\tau_2)^{n-m_2}dt, & i>1. \end{cases} \qquad (3.11)$$

The matrix G with $(i,l)^{th}$ entry is constructed as

$G_{1,l} = G_1(x_l) = \frac{1}{(n-m_2)!}\int_0^x k_2(x_l,t)t^{n-m_2}dt$ for $i=1$ and for $i = 2,...,2^{J+1}.$

$$G_{i,l}=G_i(x_l)=\frac{1}{(n-m)!}\begin{cases}0, & if\ x_l\in[0,\tau_1(i)),\\ \int_{\tau_1}^{x_l}k_2(x_l,t)(t-\tau_1(i))^{n-m_2}dt, & if\ x_l\in[\tau_1(i),\tau_2(i)),\\ \int_{\tau_1}^{x_l}k_2(x_l,t)(t-\tau_1(i))^{n-m_2}dt-2\int_{\tau_2}^{x_l}k_2(x_l,t)(t-\tau_2(i))^{n-m_2}dt, & if\ x_l\in[\tau_2(i),\tau_3(i)),\\ \int_{\tau_1}^{x_l}k_2(x_l,t)(t-\tau_1(i))^{n-m_2}dt-2\int_{\tau_2}^{x_l}k_2(x_l,t)(t-\tau_2(i))^{n-m_2}dt+\int_{\tau_3}^{x_l}k_2(x_l,t)(t-\tau_3(i))^{n-m_2}dt, & if\ x_l\in[\tau_3(i),1).\end{cases}\quad(3.12)$$

Discretize the equation (3.8) at collocation points: $x_l=\frac{(\tilde{x}_{l-1}-\tilde{x}_l)}{2}$, $l=1,2,...,2^{J+1}$ (where $\tilde{x}_c$ is the grid point given by $\tilde{x}_c=a+c\Delta t,\quad c=0,1,...,2^{J+1}$). We get $2^{J+1}\times 2^{J+1}$ linear system

$$\sum_{i=1}^{2^{J+1}}a_i\left[h_i(x_l)-E_i(x_l)-G_i(x_l)\right]=g(x_l)+\sum_{\nu=0}^{n-m_1-1}\frac{\beta_{n-m_1-\nu}}{\nu!}T_\nu(x_l)+\sum_{\nu=0}^{n-m_2-1}\frac{\beta_{n-m_2-\nu}}{\nu!}U_\nu(x_l)\ ,$$

$$a[H-E-G]=F$$

$$H_{i,l}=h_i(x_l),\ G_{i,l}=G_i(x_l),\ E_{i,l}=E_i(x_l),\ g_l=g(x_l),U_{\nu,l}=U_\nu(x_l)\ \text{and}\ T_{\nu,l}=T_\nu(x_l),$$

$$F_l=g(x_l)+\sum_{\nu=0}^{n-m_1-1}\frac{\beta_{n-m_1-\nu}}{\nu!}T_\nu(x_l)+\sum_{\nu=0}^{n-m_2-1}\frac{\beta_{n-m_2-\nu}}{\nu!}U_\nu(x_l)$$

4. Numerical Experiments

We applied the algorithm on few problems whose exact solutions are known. The obtained results are represented in the form of graphs and tables with comparisons.

Example 1: Consider the following mixed linear Fredholm-Volterra IDE [7]

$$y^{(2)}(x)+xy^{(1)}(x)-xy(x)=e^x-\sin(x)+\frac{x\cos(x)}{2}+\int_0^1\sin(x)e^{-t}y(t)dt$$
$$-\frac{1}{2}\int_0^x\cos(x)e^{-t}y(t)dt,\qquad x\in(0,1)\qquad(4.1)$$

with $y(0)=1,\ \ y^{(1)}(0)=1.$

The exact solution is $y(x)=e^x$.

The obtained $P1$, E and G matrices for $J=2$ are as follows:

$$P1=P_{1,i}(x)=\int_0^x h_i(t)dt,$$

$$P1=\frac{1}{16}\begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 \\ 1 & 3 & 5 & 7 & 7 & 5 & 3 & 1 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Similarly $P2=P_{2,i}(x)=\int_0^x P_{1,i}(t)dt.$

$$E_i(x)=\int_0^1 \sin(x)e^{-t}y(t)dt,$$

$$E=\begin{bmatrix} \frac{29}{5782} & \frac{83}{5545} & \frac{253}{10248} & \frac{537}{15784} & \frac{57}{1331} & \frac{146}{2845} & \frac{517}{8868} & \frac{372}{5747} \\ \frac{91}{23182} & \frac{164}{13999} & \frac{110}{5693} & \frac{812}{30495} & \frac{389}{11606} & \frac{55}{1379} & \frac{167}{3660} & \frac{23}{454} \\ \frac{31}{19136} & \frac{56}{11593} & \frac{203}{25458} & \frac{435}{39586} & \frac{71}{5133} & \frac{102}{6197} & \frac{179}{9506} & \frac{65}{3109} \\ \frac{55}{131733} & \frac{9}{7233} & \frac{142}{69097} & \frac{41}{14477} & \frac{35}{9818} & \frac{34}{8015} & \frac{35}{7212} & \frac{17}{3155} \\ \frac{19}{37747} & \frac{55}{36613} & \frac{13}{5247} & \frac{93}{27238} & \frac{34}{7911} & \frac{253}{49470} & \frac{12}{2051} & \frac{114}{17549} \\ \frac{30}{95971} & \frac{25}{26798} & \frac{12}{7799} & \frac{23}{10847} & \frac{137}{51329} & \frac{76}{23939} & \frac{37}{10183} & \frac{49}{12146} \\ \frac{3}{18289} & \frac{19}{38812} & \frac{15}{18578} & \frac{49}{44038} & \frac{97}{69257} & \frac{76}{45601} & \frac{19}{9965} & \frac{8}{3779} \\ \frac{8}{165517} & \frac{5}{34663} & \frac{16}{67253} & \frac{9}{27451} & \frac{8}{19385} & \frac{19}{38950} & \frac{20}{35599} & \frac{36}{57713} \end{bmatrix}$$

$$G_i(x)=\int_0^x \frac{1}{2}\cos(x)e^{-t}y(t)dt,$$

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$$G=\begin{bmatrix} \frac{6}{309647} & \frac{18}{38363} & \frac{183}{95416} & \frac{319}{69808} & \frac{150}{18119} & \frac{128}{10149} & \frac{59}{3495} & \frac{223}{10895} \\ \frac{6}{309449} & \frac{18}{38363} & \frac{183}{95416} & \frac{319}{69808} & \frac{59}{7144} & \frac{135}{11098} & \frac{105}{6886} & \frac{139}{8251} \\ \frac{6}{309647} & \frac{18}{38363} & \frac{44}{23291} & \frac{22}{5647} & \frac{17}{3018} & \frac{122}{18039} & \frac{23}{3156} & \frac{306}{42289} \\ 0 & 0 & 0 & 0 & \frac{7}{702717} & \frac{18}{80407} & \frac{83}{100233} & \frac{35}{22671} \\ \frac{6}{309647} & \frac{43}{98729} & \frac{14}{12387} & \frac{31}{18406} & \frac{10}{4821} & \frac{25}{10873} & \frac{65}{27503} & \frac{250}{109893} \\ 0 & 0 & \frac{2}{139005} & \frac{32}{102325} & \frac{31}{39617} & \frac{25}{22339} & \frac{65}{27503} & \frac{50}{109893} \\ 0 & 0 & 0 & 0 & \frac{7}{70277} & \frac{17}{81810} & \frac{17}{34315} & \frac{28}{41953} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{317123} & \frac{11}{88763} \end{bmatrix}$$

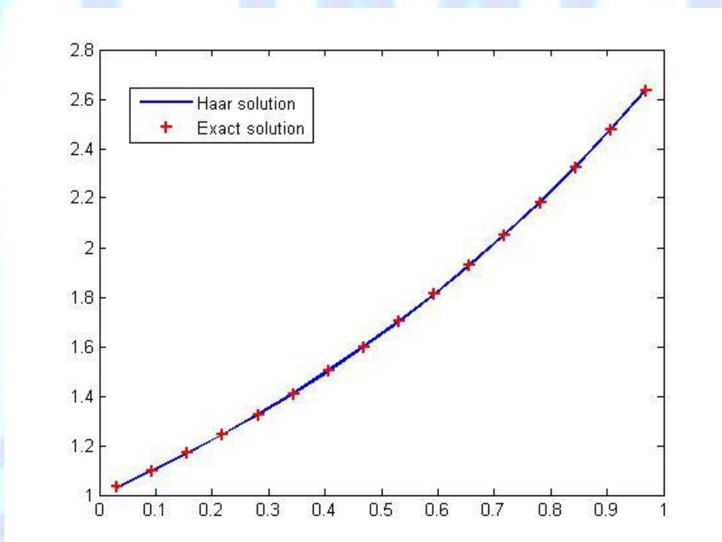

Figure 1: Comparison of Haar and exact solution with ***J=3*** for **Example 1**

Table 1: Absolute errors obtained for **Example 1**

X	Approximate solution	Exact solution	Absolute error
0.1	1.1052	1.1052	5.2497e-07
0.2	1.2214	1.2214	1.0657e-06
0.3	1.3499	1.3499	1.6219e-06
0.4	1.4918	1.4918	2.1951e-06
0.5	1.6487	1.6487	2.7897e-06
0.6	1.8221	1.8221	3.3958e-06
0.7	2.0138	2.0138	4.0003e-06
0.8	2.2255	2.2255	4.6065e-06
0.9	2.4596	2.4596	5.2215e-06

Example 2: The linear Fredholm-Volterraintegro-differential equation is [6]

$$y^{(3)}(x) = x - 2x^3 + 3x^4 + \int_0^1 xy(t)dt + \int_0^x ty(t)dt, \qquad x \in (0,1), \tag{4.2}$$

with $y(0) = 0,\ \ y^{(1)}(0) = 6,\ \ y^{(2)}(0) = -24.$

Its analytic solution is $y(x) = 6x - 12x^2.$

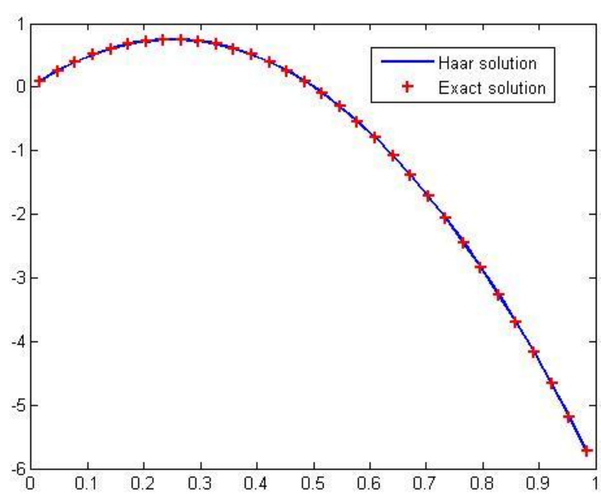

Figure 2: Comparison of Haar and exact solution with ***J=4*** for **Example 2**

Table 2: Absolute errors obtained for **Example 2**

X	Approximate solution	Exact solution	Absolute error
0.1	0.4800	0.4800	0.0000e-0
0.2	0.7200	0.7200	0.0000e-0
0.3	0.7200	0.7200	0.0000e-0
0.4	0.4800	0.4800	0.0000e-0
0.5	0.0000	0.0000	0.0000e-0
0.6	-0.7200	-0.7200	0.0000e-0
0.7	-1.6800	-1.6800	0.0000e-0
0.8	-2.8800	-2.8800	0.0000e-0
0.9	-4.3200	-4.3200	0.0000e-0

Example 3: Let us consider the mixed linear IDE [7]

$$y^{(5)}(x) - xy^{(2)}(x) + xy(x) = -e^{-x} - \frac{e^{2x}}{2} - x^2 + \frac{1}{2}\int_0^1 e^{2x+t} y(t)dt + \int_0^x xe^t y(t)dt, \quad x \in (0,1), \qquad (4.3)$$

with $y(0)=1,\ y^{(1)}(0)=-1,\ y^{(2)}(0)=1,\ y^{(3)}(0)=-1,\ y^{(4)}(0)=1.$

Exact solution is given by $y(x)=e^{-x}$.

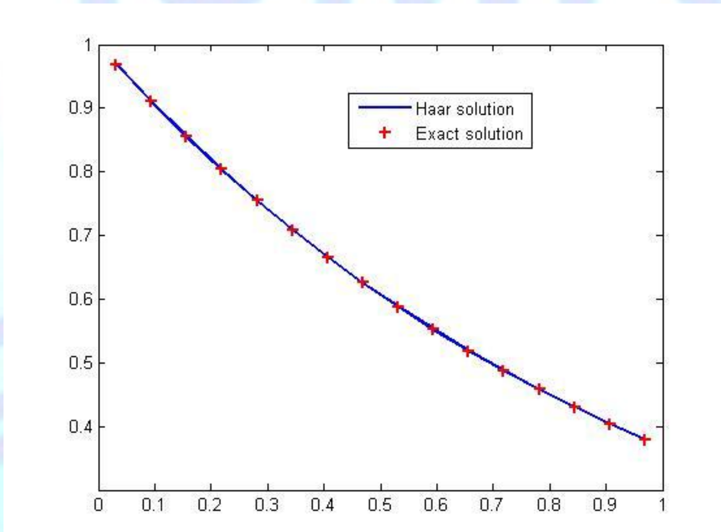

Figure 3: Comparison of Haar and exact solution with ***J=3*** for **Example 3**

Table 3: Absolute errors obtained for **Example 3**

X	Approximate solution	Exact solution	Absolute errors for $J=7$	Bessel Polynomials Method[7]
0.0	1.0000	1.0000	0.0000E-00	4.5002E-10
0.2	0.8187	0.8187	8.3156E-11	6.7190E-10
0.4	0.6703	0.6703	1.3066E-09	2.1539E-09
0.6	0.5488	0.5488	6.5030E-09	6.7558E-07
0.8	0.4493	0.4493	2.0229E-08	1.9692E-05
1.0	0.3679	0.3679	4.8681E-08	1.4536E-04

5. Conclusion

In this paper, algorithm based on Haar wavelets is introduced to solve n^{th} order mixed Fredholm-Volterraintegro-differential equations. Efficiency and applicability of the method are illustrated by considering few test problems. By the analysis of numerical experiments, we concluded that the algorithm which we introduced is an easy and effective tool for the solution of n^{th} order mixed IDEs. The accuracy of the solutions is shown by the comparison of results numerically and graphically.

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