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ON FUNCTIONS OF A SINGLE MATRIX ARGUMENT – II

Lalit Mohan Upadhyaya, Department of Mathematics, Municipal Post Graduate College, Mussoorie, Dehradun, Uttarakhand, India, 248179.

Abstract

In continuation of my previous studies [7-9], I have utilized my definition of the $U\left(a;b;-Z\right)$ function of matrix argument along with the Mathai's definitions to establish a number of results for the ${}_1F_1, {}_2F_1, {}_rF_s$ functions and the function $U(a;b;-Z)$ of matrix arguments in this paper besides giving a proof of the well-known Meijer's integral and a new proof of the confluence of Joshi and Joshi [2].

Keywords: ${}_1F_1, {}_2F_1, {}_rF_s$ and $U(a;b;-Z)$ functions, matrix argument, matrix transform.

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1. Introduction

This paper carries ahead my earlier study [9] of functions of single matrix arguments in which a number of new results have been established for the ${}_1F_1, {}_2F_1, {}_rF_s$ functions and the function $U(a;b;-Z)$ of matrix arguments. Since functions of single matrix argument play a vital role in many applications of mathematics e.g. in statistical distribution theory, multivariate statistical analysis, econometrics, etc. to name a few, the results developed in the present paper would find immediate applications in these and allied areas of current research interest. The introductory part of the paper deals with the explanations of the notations used in the paper and the preliminary results and definitions, which are used to develop the results in the next part of the paper. The main results of the paper have been established in the second section of the paper. For the development of the results in this paper, we state that all the matrices appearing in this paper are real symmetric and positive definite with order $(p\times p)$. $A>0$ will mean that the matrix A is positive definite, $A^{1/2}$ will represent the symmetric square root of A. While integrating over matrices $\int_X f(X)dX$ represents integral over X of the scalar function $f(X)$. $\mathrm{Re}(.)$ denotes the real part of $(.)$.

We begin with quoting some preliminary results and definitions, which occur in the literature. Mathai [3] in 1978 defined the matrix transform (M- transform) of a function $f(X)$ of a $(p\times p)$ real symmetric positive definite matrix X as follows:

$$M_f(\rho)=\int_{X>0}|X|^{\rho-(p+1)/2} f(X)dX \tag{1.1}$$

for $X>0$ and $\mathrm{Re}(\rho)>(p-1)/2$ whenever $M_f(s)$ exists.

We will use the following results and definitions at various places in this paper.

Theorem 1.1: Mathai [4] (eq. 2.24, p.23)- Let X and Y be $(p\times p)$ symmetric matrices of functionally independent real variables and A a $(p\times p)$ non singular matrix of constants. Then,

$$Y=AXA'\Rightarrow dY=|A|^{p+1}dX \tag{1.2}$$

and

$$Y=aX\Rightarrow dY=a^{p(p+1)/2}dX \tag{1.3}$$

where a is a scalar quantity.

Theorem 1.2: Gamma integral (Mathai [5], eq.(2.1.3), p.33, eq.(2.1.2), p. 32)-

$$\int_{X>0}|X|^{\alpha-(p+1)/2}e^{-tr(BX)}dX=|B|^{-\alpha}\Gamma_p(\alpha) \tag{1.4}$$

for $\mathrm{Re}(\alpha)>(p-1)/2$ where,

$$\Gamma_p(\alpha)=\pi^{p(p-1)/4}\Gamma(\alpha)\Gamma\left(\alpha-\tfrac{1}{2}\right)\cdots\Gamma\left(\alpha-\tfrac{p-1}{2}\right) \tag{1.5}$$

for $\mathrm{Re}(\alpha)>(p-1)/2$ and $tr(X)$ denotes the trace of the matrix X.

Theorem 1.3: Type-1 Beta Integral (Mathai [5], eq. (2.2.2), p.34)-

$$B_p(\alpha,\beta)=\int_{0<X<I}|X|^{\alpha-(p+1)/2}|I-X|^{\beta-(p+1)/2}dX=\frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} \tag{1.6}$$

for $\mathrm{Re}(\alpha) > (p-1)/2, \mathrm{Re}(\beta) > (p-1)/2$.

Theorem 1.4: (Mathai [5], eq. (2.3.13), p.42)-

$$ {}_1F_1(a;c;-X) = \frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)} \int_0^I |Y|^{a-(p+1)/2} |I-Y|^{c-a-(p+1)/2} e^{-tr(XY)} dY \tag{1.7} $$

for $\mathrm{Re}(a, c-a) > (p-1)/2$.

Theorem 1.5: (Mathai [5], eq.(2.3.6) , p.38)-

$$ {}_2F_1(a,b;c;-X) = \frac{\Gamma_p(c)}{\Gamma_p(a)\Gamma_p(c-a)} \int_0^I |Y|^{a-(p+1)/2} |I-Y|^{c-a-(p+1)/2} |I+XY|^{-b} dY \tag{1.8} $$

for $0 < X < I$ and $\mathrm{Re}(a, c-a) > (p-1)/2$.

Theorem 1.6: Type-2 Beta Integral: (Mathai [5], eq.(2.2.4), p.36 and eq.(2.2.2), p.34)- For a real symmetric positive definite matrix Y ,

$$ B_p(\alpha, \beta) = \int_{Y>0} |Y|^{\alpha-(p+1)/2} |I+Y|^{-(\alpha+\beta)} dY = \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)} \tag{1.9} $$

where $\mathrm{Re}(\alpha, \beta) > (p-1)/2$.

Theorem 1.7: (Mathai [5], eq.(2.3.11), p.41)- Let $f(X)$ be a symmetric function in the sense $f(AX) = f(XA)$ where X and A are $(p \times p)$ real symmetric positive definite matrices. Under interchangeability of limits and integrals

$$ M\left[\lim_{\alpha\to\infty} f\left(\frac{X}{\alpha}\right)\right] = \lim_{\alpha\to\infty} \alpha^{p\rho} M\left[f(X)\right] \tag{1.10} $$

where $M[.]$ denotes the M-transform with respect to the parameter ρ.

Theorem 1.8: (Mathai[5], eq.(2.3.10), p.40) For finite ρ

$$\lim_{\alpha\to\infty} \alpha^{p\rho} \frac{\Gamma_p(\alpha-\rho)}{\Gamma_p(\alpha)} = 1 \tag{1.11}$$

Definition 1.1: The ${}_rF_s$ function of matrix arguments

$${}_rF_s = {}_rF_s\left(a_1,\cdots,a_r;b_1,\cdots,b_s;-X\right)$$

is defined as that class of functions which has the following matrix transform,

$$M\left({}_rF_s\right) = \int_{X>0} |X|^{\rho-(p+1)/2} {}_rF_s\left(a_1,\cdots,a_r;b_1,\cdots,b_s;-X\right)dX$$

$$= \frac{\left\{\prod_{k=1}^{s}\Gamma_p\left(b_k\right)\right\}\left\{\prod_{m=1}^{r}\Gamma_p\left(a_m-\rho\right)\right\}}{\left\{\prod_{k=1}^{s}\Gamma_p\left(b_k-\rho\right)\right\}\left\{\prod_{m=1}^{r}\Gamma_p\left(a_m\right)\right\}}\Gamma_p\left(\rho\right) \tag{1.12}$$

for $\mathrm{Re}(\rho, a_m-\rho, b_k-\rho) > (p-1)/2$ where, $m=1,\cdots,r; k=1,\cdots,s$.

Definition 1.2: (Upadhyaya and Dhami [9, eq.(1.11), p.67]) The confluent hypergeometric function $U(a;b;-Z)$ of second kind with matrix argument is defined as that class of functions for which the M-transform is the following:

$$M[U(a;b;-Z)] = \int_{Z>0} |Z|^{\rho-(p+1)/2} U(a;b;-Z)dZ = \frac{\Gamma_p(a-\rho)\Gamma_p[(p+1)/2-b+\rho]\Gamma_p(\rho)}{\Gamma_p(a)\Gamma_p[(p+1)/2+a-b]} \tag{1.13}$$

for $\mathrm{Re}(a-\rho,(p+1)/2-b+\rho,\rho) > (p-1)/2$.

2. Results

Now I shall deduce the main results of the paper in the form of the following theorems:

Theorem 2.1:

$$\int_{T>0} e^{-tr(ST)} U(a;c;-T)|T|^{b-(p+1)/2}\, dT$$

$$= \frac{\Gamma_p(b)\Gamma_p[(p+1)/2+b-c]}{\Gamma_p[(p+1)/2+a+b-c]} \times {}_2F_1[b,(p+1)/2+b-c;(p+1)/2+a+b-c;-(S-I)] \quad (2.1)$$

for $\mathrm{Re}[b,(p+1)/2+b-c,(p+1)/2+a+b-c] > (p-1)/2$.

Proof: The application of the theorem (1.5) to the ${}_2F_1$ on the right side of eq.(2.1) yields

$${}_2F_1[b,(p+1)/2+b-c;(p+1)/2+a+b-c;-(S-I)]$$

$$= \frac{\Gamma_p[(p+1)/2+a+b-c]}{\Gamma_p(b)\Gamma_p[(p+1)/2+a-c]} \int_0^I |Y|^{b-(p+1)/2} |I-Y|^{a-b-(p+1)/2} \times$$

$$\left| I + (I-Y)^{-1/2} Y^{1/2} S Y^{1/2} (I-Y)^{-1/2} \right|^{c-b-(p+1)/2} dY \quad (2.2)$$

which is to be substituted on the right side of eq.(2.1) for the ${}_2F_1$ function. Then we take the M-transform of the modified form of the right side of eq.(2.1) with respect to the variable S and the parameter ρ, consequently apply the transformation $S_1 = (I-Y)^{-1/2} Y^{1/2} S Y^{1/2} (I-Y)^{-1/2}$ and integrate out S_1 by using a type-2 Beta integral (theorem (1.6)) and Y by using a type-1 Beta integral (theorem (1.3)) to achieve

$$\frac{\Gamma_p[(p+1)/2+b-c-\rho]\Gamma_p(a-b+\rho)\Gamma_p(b-\rho)\Gamma_p(\rho)}{\Gamma_p(a)\Gamma_p[(p+1)/2+a-c]} \quad (2.3)$$

The same result is obtained by taking the M-transform of the left side of eq.(2.1) with respect to the variable S and the parameter ρ and subsequently using the definition (1.2) in it.

Theorem 2.2:

$$\int_{T>0} e^{-tr(ST)} U(a;c;-T)|T|^{b-(p+1)/2}\, dT$$

$$=\frac{\Gamma_p(b)\Gamma_p[(p+1)/2+b-c]|S|^{-b}}{\Gamma_p[(p+1)/2+a+b-c]}\times {}_2F_1[a,b;(p+1)/2+a+b-c;-(S^{-1}-I)] \qquad (2.4)$$

for $\mathrm{Re}[b,(p+1)/2+b-c,(p+1)/2+a+b-c]>(p-1)/2$.

Proof: This theorem follows in the same manner as the previous theorem, the ${}_2F_1$ function on the right side of eq.(2.4) is to be replaced by the following expression which is achieved by the application of the theorem (1.5) to the ${}_2F_1$ function on the right side of eq.(2.4):

$${}_2F_1[a,b;(p+1)/2+a+b-c;-(S^{-1}-I)]$$

$$=\frac{\Gamma_p[(p+1)/2+a+b-c]}{\Gamma_p(a)\Gamma_p[(p+1)/2+b-c]}\int_0^I |Y|^{a-(p+1)/2}|I-Y|^{(p+1)/2-c-(p+1)/2}\times$$

$$\left|I+(I-Y)^{-1/2}Y^{1/2}S^{-1}Y^{1/2}(I-Y)^{-1/2}\right|^{-b}dY \qquad (2.5)$$

Theorem 2.3: Meijer's Integral:

$$U(a;d;-X)=\frac{|X|^{b-a}}{\Gamma_p(b)}\int_{T>0}e^{-tr(XT)}|T|^{b-(p+1)/2}\,{}_2F_1[a,(p+1)/2+a-d;b;-T]dT \qquad (2.6)$$

for $\mathrm{Re}(b)>(p-1)/2$.

Proof: Taking the M-transform of the right side of eq.(2.6) with respect to the

variable X and the parameter ρ and integrating out X by using a Gamma integral (theorem (1.2)) and then writing the M-transform of the ${}_2F_1$ function with parameter $a-\rho$ we obtain $M[U(a;b;-Z)]$ as given by the definition (1.2).

Theorem 2.4:

$$\int_{X>0}e^{-tr(AX)}|X|^{s-(p+1)/2}U(b;d;-X^{1/2}KX^{1/2})dX$$

$$=\frac{\Gamma_p(s)\Gamma_p[(p+1)/2+s-d]}{\Gamma_p[(p+1)/2+b+s-d]}|A|^{-s}\ {}_2F_1[b,s;(p+1)/2+b+s-d;-(A^{-1/2}KA^{-1/2}-I)] \quad (2.7)$$

for $\mathrm{Re}[s,(p+1)/2+s-d,(p+1)/2+b+s-d]>(p-1)/2$.

Proof: We apply the theorem (1.5) to replace the ${}_2F_1$ function on the right side of eq.(2.7) by the following integral:

$${}_2F_1[b,s;(p+1)/2+b+s-d;-(A^{-1/2}KA^{-1/2}-I)]$$

$$=\frac{\Gamma_p[(p+1)/2+b+s-d]}{\Gamma_p(b)\Gamma_p[(p+1)/2+s-d]}\int_0^I |Y|^{b-(p+1)/2}|I-Y|^{(p+1)/2-d-(p+1)/2}\times$$

$$\left|I+(I-Y)^{-1/2}Y^{1/2}A^{-1/2}KA^{-1/2}Y^{1/2}(I-Y)^{-1/2}\right|^{-s}dY \quad (2.8)$$

then taking the M-transform of the right side of eq.(2.7) with respect to the variable K and the parameter ρ, which under the use of the transformation

$$K_1=(I-Y)^{-1/2}Y^{1/2}A^{-1/2}KA^{-1/2}Y^{1/2}(I-Y)^{-1/2}$$

and integrating out K_1 by employing a type-2 Beta integral (theorem (1.6)) and Y by a type-1 Beta integral (theorem (1.3)) lends

$$|A|^{\rho-s}\frac{\Gamma_p[(p+1)/2-d+\rho]\Gamma_p(s-\rho)\Gamma_p(b-\rho)\Gamma_p(\rho)}{\Gamma_p(b)\Gamma_p[(p+1)/2-d+b]} \quad (2.9)$$

This result is also achieved by taking the M-transform of the left side of eq.(2.7) with respect to the variable K and the parameter ρ under the transformation $K_2=X^{1/2}KX^{1/2}$ and using the definition (1.2) and integrating out X by using a Gamma integral (theorem (1.2)).

It is interesting to note that the left side of eq.(2.7) may be treated as the M-transform of the function $e^{-tr(AX)}U(b;d;-X^{1/2}KX^{1/2})$ with respect to the variable X and the parameter s and on the right side it depends on s, the parameter of the transform!

Theorem 2.5: The Confluence of Joshi and Joshi [2]:

$$B_{c-(p+1)/2}(Z) = \lim_{a\to\infty}\Gamma_p\left[a-c+(p+1)/2\right]U\left(a;c;-\frac{Z}{a}\right) \tag{2.10}$$

Proof: The above confluence has been stated by Joshi and Joshi [2, eq.(20), p.633]. We provide here an alternative proof for this result by using the M-transform technique.

In the left side of eq.(2.10), $B_{c-(p+1)/2}(Z)$ represents the Bessel function of the second kind with matrix argument. This notation was earlier used by Herz [1,p.506]. We use his results to establish the above confluence.

Eq. (5.1'), p.506, Herz[1] is

$$\begin{aligned}{}_lG_1\left[\alpha_1,\cdots,\alpha_l;\delta+(p+1)/2;Z\right] &= \Gamma_p\left[\delta+(p+1)/2\right]\int_{\Lambda>0} e^{-tr(\Lambda)}\times \\ &{}_lF_0(\alpha_1,\cdots,\alpha_l;\ ;-\Lambda^{-1}Z)\left|\Lambda\right|^{-\delta-(p+1)/2}d\Lambda;\end{aligned} \tag{2.11}$$

and according to him [1, p.506]

$$B_\delta(Z) = \frac{{}_0G_1\left[\ ;\delta+(p+1)/2;Z\right]}{\Gamma_p\left[\delta+(p+1)/2\right]} \tag{2.12}$$

which, on putting $l=0$ in eq.(2.11), yields

$$B_\delta(Z) = \int_{\Lambda>0} e^{-tr(\Lambda+\Lambda^{-1}Z)}\left|\Lambda\right|^{-\delta-(p+1)/2}d\Lambda \tag{2.13}$$

Now taking the M-transform of the function $B_\delta(Z)$ with respect to the variable Z and the parameter ρ we have,

$$M\left[B_\delta(Z)\right] = \int_{Z>0}\left|Z\right|^{\rho-(p+1)/2}B_\delta(Z)dZ \tag{2.14}$$

which on evaluation with the help of eq.(2.13), first on integrating out Z and then Λ by utilizing a Gamma integral (theorem (1.2)) yields,

$$M\left[B_\delta(Z)\right] = \Gamma_p(\rho)\Gamma_p(\rho-\delta) \tag{2.15}$$

from which the M-transform of the left side of eq.(2.10) can be had by putting $\delta = c-(p+1)/2$. Now taking the M-transform of the right side of eq.(2.10) with respect to the variable Z and the parameter ρ we note

$$M\left[\lim_{a\to\infty}\Gamma_p\left[a-c+(p+1)/2\right]U\left(a;c;-\frac{Z}{a}\right)\right] = \lim_{a\to\infty} a^{m\rho} M\left[\Gamma_p\left[a-c+(p+1)/2\right]U\left(a;c;-Z\right)\right] \tag{2.16}$$

from theorem (1.7), which on utilizing the definition (1.2) and the theorem (1.8) ultimately produces $\Gamma_p(\rho)\Gamma_p\left[\rho-c+(p+1)/2\right]$ which is the M-transform of the left side of eq.(2.10) (from eq.(2.15) for $\delta = c-(p+1)/2$), thus establishing the theorem.

Theorem 2.6:

$$\int_{T>0} e^{-tr(ST)}|T|^{b-(p+1)/2}\,{}_1F_1(a;c;-KT)dT = \Gamma_p(b)|S+K|^{-b}\,{}_2F_1[c-a,b;c;K(S+K)^{-1}] \tag{2.17}$$

for $\mathrm{Re}(b) > (p-1)/2$.

Proof: This result can be had by replacing the ${}_1F_1$ function on the left side of eq.(2.17) by the use of the theorem (1.4) consequently applying the transformation $Y_1 = I - Y$ and observing that

$$|S+(I-Y)K| = |S+K|\left|I-Y_1K(S+K)^{-1}\right|$$

and finally using the theorem (1.5).

Letting $K \to I$ in the theorem (2.6) we have the following interesting result:

$$\int_{T>0} e^{-tr(ST)}|T|^{b-(p+1)/2}\,{}_1F_1(a;c;-T)dT = \Gamma_p(b)|S+I|^{-b}\,{}_2F_1[c-a,b;c;(S+I)^{-1}] \tag{2.18}$$

for $\mathrm{Re}(b) > (p-1)/2$.

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In a similar manner, we can establish the following results:

Theorem 2.7:

$$\int_{T>0} e^{-tr(ST)} |T|^{b-(p+1)/2} {}_1F_1(a;b;-T^{\frac{1}{2}}KT^{\frac{1}{2}})dT = \Gamma_p(b)|S|^{-b} {}_1F_0[a;\ ;-S^{-\frac{1}{2}}KS^{-\frac{1}{2}}] \tag{2.19}$$

for $\mathrm{Re}(b) > (p-1)/2$.

Theorem 2.8:

$$\int_{T>0} e^{-tr(ST)} |T|^{a-(p+1)/2} {}_0F_1(\ ;b;-T^{\frac{1}{2}}KT^{\frac{1}{2}})dT = \Gamma_p(a)|S|^{-a} {}_1F_1[a;b;-S^{-\frac{1}{2}}KS^{-\frac{1}{2}}] \tag{2.20}$$

for $\mathrm{Re}(a) > (p-1)/2$.

Theorem 2.9:

$$\int_{T>0} e^{-tr(ST)} |T|^{a-(p+1)/2} {}_0F_1(\ ;a;-T^{\frac{1}{2}}KT^{\frac{1}{2}})dT = \Gamma_p(a)|S|^{-a} e^{-tr(KS^{-1})} \tag{2.21}$$

for $\mathrm{Re}(a) > (p-1)/2$.

Theorem 2.10:

$$\int_{T>0} e^{-tr(AX)} |X|^{s-(p+1)/2} {}_1F_1(b;d;-X^{\frac{1}{2}}KX^{\frac{1}{2}})dX = \Gamma_p(s)|A|^{-s} {}_2F_1[b,s;d;-A^{-\frac{1}{2}}KA^{-\frac{1}{2}}] \tag{2.22}$$

for $\mathrm{Re}(s) > (p-1)/2$.

Theorem 2.11:

$$\int_{T>0} e^{-tr(X)} |X|^{s-(p+1)/2} {}_0F_1(\ ;b;-X^{\frac{1}{2}}YX^{\frac{1}{2}})dX = \Gamma_p(s) {}_1F_1(s;b;-Y) \tag{2.23}$$

for $\mathrm{Re}(s) > (p-1)/2$.

Theorem 2.12:

$$\int_{T>0} e^{-tr(KX)} \left|X\right|^{\rho-(p+1)/2} {}_rF_s(a_1,\cdots,a_r;b_1,\cdots,b_s;-X^{1/2}YX^{1/2})dX$$

$$=\Gamma_p(\rho)\left|K\right|^{-\rho} {}_{r+1}F_s(a_1,\cdots,a_r,\rho;b_1,\cdots,b_s;-K^{-1/2}YK^{-1/2}) \quad (2.24)$$

for $\mathrm{Re}(\rho) > (p-1)/2$.

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