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Existence of Solution to Fractional Order Integral Equations.

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ABSTRACT:
In this article, we discuss the existence solution to fractional order quadratic volterra integral equation in R_+. Hybrid fixed point theorem is used here to obtain the existence result.

KEYWORDS: Banach algebras, Existence result, Nonlinear equations, Quadratic integral equation.

INTRODUCTION:

Fractional calculus developed only as the theoretical field of mathematics. Fractional differential equation plays an important role in the study of various physical chemical and biological phenomenon's many researchers are attracted from the field of theory methods and application of fractional differential equation. The research have developed are various method to obtain of techniques to obtained approximate solutions of both linear and nonlinear fractional differential and integral equations The theory of integral equations of fractional order has recently received a lot of attention and constitutes a significant branch of nonlinear analysis. Numerous research papers and monographs devoted to differential and integral equations of fractional order have appeared (see [2-3, 8-10] These papers contain various types of existence results for equations of fractional order. In this paper; we study the existence results of the following nonlinear quadratic volterra integral equation of fractional order.

Let $\beta \in (0,1)$ and R denote the real numbers whereas R_+ be the set of nonnegative numbers i.e $R_+ = [0,\infty) \subset R$.

Consider thefractional order nonlinear quadratic volterra integral equation :

$$x(t) = [f(x(t))]\left(q(t) + \frac{1}{\Gamma(\beta)}\int_0^t \frac{g(s,x(s))}{(t-s)^{1-\beta}}\,ds\right) \qquad \forall t \in R_+ \qquad 1.1$$

Where $q : R_+ \to R$, $f\colon R_+ \to R$, $g(t,x) = g : R_+ \times R \to R$

By a solution of the (1.1) we mean a function $x \in BC(R_+, R)$ that satisfies (1.1) on R_+ .

Where $BC(R_+, R)$ is the space of continuous and bounded real-valued functions defined on R_+.

In this paper, we prove the existence of solution for (1.1) employing a classical hybrid fixed point theorem of Dhage [4]. In the next section, we collect some preliminary definitions and auxiliary results that will be used in the follows.

1. Preliminaries:

Let $X = BC(R_+, R)$ be Banach algebra with norm $\|\cdot\|$ and let Ω be a subset of X. Let a mapping $A\colon X \to X$ be an operator and consider the following operator equation in X, namely,

$x(t)=(Ax)(t) \quad t\in R_+ \ (2.1)$

Below we give different characterizations of the solutions for operator equation (2.1) on R_+.

We need the following definitions in the sequel.

Definition 2.1: Let *X* be a Banach space. A mapping $A: X \to X$ is called Lipschitz if there is a constant $\alpha > 0$ such that $\|Ax - Ay\| \le \alpha\|x-y\|$ for all $x, y \in X$. If $\alpha < 1$, then *A* is called a **contraction** on *X* with the contraction constant α.

Definition 2.2: (Dugundji and Granas [8]). An operator A on a Banach space X into itself is called Compact if for any bounded subset S of X, A(S) is a relatively compact subset of X. If A is continuous and compact, then it is called completely continuous on X.

Let *X* be a Banach space with the norm $\|.\|$ and Let $A: X \to X$ be an operator (in general nonlinear). Then *A* is called

(i) Compact if $A(X)$ is relatively compact subset of X;

(ii) totally bounded if $A(S)$ is a totally bounded subset of X for any bounded subset S of X

(iii) Completely continuous if it is continuous and totally bounded operator on *X*.

It is clear that every compact operator is totally bounded but the converse need not be true.

The solutions of (1.1) in the space $BC(R_+,R)$ of continuous and bounded real-valued functions defined on R_+. Define a standard supremum norm $\|.\|$ and a multiplication "." in $BC(R_+,R)$ by

$\|x\| = \sup\{|x(t)| : t\in R_+\}$, (2.2)

$(xy)(t) = x(t)y(t)$, $t \in R_+$. (2.3)

Clear, $BC(R_+,R)$ becomes a Banach space with respect to the above norm and the multiplication in it.

By $L^1(R_+,R)$ we denote the space of Lebesgue integrable functions on R_+ with the norm $\|.\|_{L^1}$ defined by $\|x\|_{L^1} = \int_0^\infty |x(t)|dt$ (2.4)

Denote by $L^1(a,b)$ be the space of Lebesgue integrable functions on the interval (a, b),which is equipped with the standard norm. Let $x\in L^1(a,b)$ and let $\beta > 0$ be a fixed number.

Definition 2.4: The left sided Riemann-Liouville fractional integral [10, 12, 18] of order β of real function f is defined as $I_{a^+}^{\beta} f(x) = \frac{1}{\Gamma(\beta)}\int_a^x \frac{f(t)}{(x-t)^{1-\beta}}dt \ \ \beta>0, x>a$

Definition 2.5: The Riemann-Liouville fractional integral of order β of the function $x(t)$ is defined by the formula: $I^{\beta} f(t) = \frac{1}{\Gamma(\beta)}\int_0^t \frac{f(s)}{(t-s)^{1-\beta}}ds$, $t\in(a,b)$ (2.5)

Where $\Gamma(\beta)$ denote the gamma function.

It may be shown that the fractional integral operator I^{β} transforms the space $L^1(a,b)$ into itself and has some other properties (see [12-19])

Theorem 2.1: (Arzela-Ascoli theorem) If every uniformly bounded and equi-continuous sequence $\{f_n\}$ of functions in $C(J,R)$, then it has a convergent subsequence.

Theorem 2.2: A metric space X is compact iff every sequence in X has a convergent subsequence.

We employ a hybrid fixed point theorem of Dhage [4] for proving the existence result.

Theorem 2.3 :(Dhage [4]). Let S be a closed-convex and bounded subset of the Banach space X and let $A, B : S \to S$ be two operators satisfying:

(a) A is Lipschitz with the Lipschitz constant k,
(b) B is completely continuous,
(c) $AxBx \in S$ for all $x \in S$,and
(d) $Mk < 1$ Where $M = \|B(S)\| = \sup\{\|Bx\| : x \in S\}$.

Then the operator equation $AxBx = x$ has a solution and the set of all solutions is compact in S.

2. EXISTENCE RESULTS:

Definition 3.1. A mapping $g : R_+ \times R \to R$ is said to be Caratheodory if

1. $t \mapsto g(t,x)$ is measurable for all $x \in R$, and
2. $t \mapsto g(t,x)$ is continuous almost everywhere for $t \in R_+$

Again a caratheodory function *g* is called L^1-Caratheodory if

3. for each real number $r > 0$ there exists a function $h_r \in L^1(R_+, R)$ such that $|g(t,x)| \le h_r(t)$ a.e. $t \in R_+$ for all $x \in R$ with $|x| \le r$.

Finally, a Caratheodory function $g(t,x)$ is called L^1_R-Caratheodory if

4. there exist a function $h \in L^1(R_+, R)$ such that $|g(t,x)| \le h(t)$ a.e. $t \in R_+$ for all $x \in R$

For convenience, the function *h* is referred to as a bound function of *g.*

We consider the following set of hypotheses in the sequel.

H$_1$: The function $f: R_+ \to R$ is continuous and bounded with bound $F = sup|f(x(t))|$ there exists a bounded function $l : R_+ \to R_+$ with bound L Satisfying $|f(x(t)) - f(y(t))| \le l(t)|x-y|$ $\quad for\ all\ t \in R_+\ and\ x, y \in R.$

H$_2$: $q : R_+ = [0, +\infty) \to R$ is continuous function on R_+; also $\lim_{t\to\infty} q(t) = 0$

H$_3$:The function $g(t,x) = g : R_+ \times R \to R$ satisfy caratheodory condition (i.e.measrable in *t* for all $x \in R$ and continuous in x for all $t \in R_+$) and there exist function $h \in L^1(R_+, R)$ Such that $g(t,x) \le h(t) \forall (t,x) \in R_+ \times R.$

In what follows we will assume additionally that the following conditions satisfied.

H$_4$: The uniform continuous function $v : R_+ \to R_+$ defined by the formulas $v(t) = \int_0^t \frac{h(s)}{(t-s)^{1-\beta}} ds$ is bounded on R_+ and vanish at infinity, that is, $\lim_{t\to\infty} v(t) = 0$.

Remark 3.1: Note that if the hypothesis H_2 and H_3 hold, then there exist constants $K_1 > 0$ and $K_2 > 0$ such that: $K_1 = \sup\{q(t) : t \in R_+\}$, $K_2 = \sup_{t\geq 0} \frac{1}{\Gamma(\beta)} \int_0^t \frac{h(s)}{(t-s)^{1-\beta}} ds$

Theorem 3.1: Suppose that the hypotheses H_1 –H_5 are hold. Furthermore if $L(K_1 + K_2) < 1$**,** where K_1 and K_2 are defined remark 3.1, Then (1.1) has a solution in the space $BC(R_+, R)$.

Proof: By a solution of the (1.1) we mean a continuous function $x : R_+ \to R$ that satisfies (1.1) on R_+.

Let $X = BC(R_+, R)$ be Banach Algebras of all continuous and bounded real valued function on R_+ with the norm $\|x\| = \sup_{t\in R_+} |x(t)|$ (3.1)

We shall obtain the solution of (1.1) under some suitable conditions on the functions involved in (1.1).

Consider the closed ball $B_r[0]$ in X centered at origin 0 and of radius r, where $r = F(K_1 + K_2) > 0$

Let's define the operators A and B on $B_r(0)$ by,

$$Ax(t) = f(x(t)) \tag{3.2}$$

$$Bx(t) = q(t) + \frac{1}{\Gamma(\beta)} \int_0^t \frac{g(s,x(s))}{(t-s)^{1-\beta}} ds \tag{3.3}$$

$$for\ all\ t \epsilon R_+$$

Since the hypotheses H_1 holds, the mapping *A* is well defined and the function *Ax* is continuous and bounded on R_+. Again the function *q* is continuous on R_+, the function *Bx* is also continuous and bounded in view of hypotheses H_2.

Therefore *A* and *B* define the operators $A, B : B_r[0] \to X$ .we shall show that *A* and *B* satisfy all the requirements of theorem 2.3 on $B_r[0]$.

Step I: Firstly, we show that *A* is Lipschitz on $B_r[0]$ .let $x, y \in B_r[0]$ be arbitrary, and then by hypothesis H_1

$$|A(x(t)) - A(y(t))| = |f(x(t)) - f(y(t))|$$
$$\leq l(t)|x - y|$$
$$\leq L\|X - Y\| \tag{3.4}$$

$$for\ all\ t\ \in R_+\ taking\ superimum\ over\ t.$$

$$\|A(x) - A(y)\| \leq L\|X - Y\|$$

for all $x, y\ \in B_r(0)$. (3.5)

This shows that A is Lipschitiz operator on $B_r(0)$ with Lipschitz constant L.

Step II: Secondly, we show that *B* is completely continuous operator on $B_r[0]$.

Firstly we show that *B* is continuous on $B_r[0]$**.**

Case I: Suppose that $t \geq T$, there exist $T > 0$ and let us fix arbitrary $\varepsilon > 0$ and take $x, y \in B_r[0]$ such that $\|x - y\| \leq \varepsilon$. Then

$$|(Bx)(t) - (By)(t)| \leq \frac{1}{\Gamma(\beta)} \int_0^t \frac{|g(s, x(s))|}{(t-s)^{1-\beta}} ds + \frac{1}{\Gamma(\beta)} \int_0^t \frac{|g(s, y(s))|}{(t-s)^{1-\beta}} ds$$

$$\leq \frac{2}{\Gamma(\beta)} \int_0^t \frac{h(s)}{(t-s)^{1-\beta}} ds$$

$$\leq \frac{2v(t)}{\Gamma(\beta)} \qquad for\ all\ t\ \geq T \qquad (3.6)$$

There exists T>0 s.t. $v(t) \leq \Gamma(\beta)\varepsilon.\frac{1}{2}$

$$\left|(Bx)(t)-(By)(t)\right| \leq \varepsilon \ (3.7)$$

Case II: Further, let us assume that $t \in [0,T]$, then evaluating similarly to above we obtain the following estimate

$$|(Bx)(t)-(By)(t)| \leq \left| \frac{1}{\Gamma(\beta)} \int_0^T \frac{g(s,x(s))}{(t-s)^{1-\beta}} ds - \frac{1}{\Gamma(\beta)} \int_0^T \frac{g(s,y(s))}{(t-s)^{1-\beta}} ds \right|$$

$$|(Bx)(t)-(By)(t)| \leq \frac{1}{\Gamma(\beta)} \int_0^T \frac{|g(s,x(s))-g(s,y(s))|}{(t-s)^{1-\beta}} ds$$

$$|(Bx)(t)-(By)(t)| \leq \frac{1}{\Gamma(\beta)} \int_0^T \frac{\omega_r^T(g,\epsilon)}{(t-s)^{1-\beta}} ds$$

$$|(Bx)(t)-(By)(t)| \leq \frac{T^\beta}{\Gamma(\beta)\,\beta} \omega_r^T(g,\epsilon) ds$$

$$|(Bx)(t)-(By)(t)| \leq \frac{T^\beta}{\Gamma(\beta+1)} \omega_r^T(g,\epsilon) ds \qquad (3.8)$$

Where $\omega_r^T(g,\varepsilon) = \sup\{|g(s,x)-g(s,y)| : s \in [0,T]; x,y \in [-r,r], |x-y| \leq \varepsilon\}$

Therefore, from the uniform continuity of the function $g(t,x)$ on the set $[0,T]\times[-r,r]$ .we derive that $\omega_r^T(g,\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now combining the case I and II, we conclude that the operator B is continuous operator on closed ball $B_r[0]$ in to itself.

Step III: Next we show that B is compact on $B_r[0]$.

(A) First prove that every sequence $\{Bx_n\}$ in $B(B_r[0])$ has a uniformly bounded sequence in $B(B_r[0])$. Now by hypothesis

$$|Bx_n(t)| \leq |q(t)| + \frac{1}{\Gamma(\beta)} \int_0^t \frac{|g(s,x_n(s)|}{(t-s)^{1-\beta}} ds$$

$$\leq |q(t)| + \frac{1}{\Gamma(\beta)} \int_0^t \frac{h(s)}{(t-s)^{1-\beta}} ds \leq |q(t)| + \frac{v(t)}{\Gamma(\beta)}$$

$$\leq K_1 + K_2 \qquad \text{for all } t \in R_+ \qquad (3.9)$$

Taking supremum over t, we obtain $\|Bx_n\| \leq K_1 + K_2$ for all $n \in N$.

This shows that $\{Bx_n\}$ is a uniformly bounded sequence in $B(B_r[0])$.

(B) Now we proceed to show that sequence $\{Bx_n\}$ is also equicontinuous.

Let $\varepsilon > 0$ be given. Since $\lim_{t\to\infty} q(t) = 0$, there is constant $T > 0$ such that $|q(t)| < \frac{\varepsilon}{2}$ for all $t \geq T$

Case I: If $t_1, t_2 \in [0, T]$, then we have

$$|Bx_n(t_2) - Bx_n(t_1)| \le \left| q(t_2) + \frac{1}{\Gamma(\beta)} \int_0^{t_2} \frac{g(s, x_n(s))}{(t_2 - s)^{1-\beta}} ds - q(t_1) - \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{g(s, x_n(s))}{(t_1 - s)^{1-\beta}} ds \right|$$

$$\le |q(t_2) - q(t_1)| + \left| \frac{1}{\Gamma(\beta)} \int_0^{t_2} \frac{g(s, x_n(s))}{(t_2 - s)^{1-\beta}} ds - \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{g(s, x_n(s))}{(t_1 - s)^{1-\beta}} ds \right|$$

$$\le |q(t_2) - q(t_1)| + \frac{1}{\Gamma(\beta)} \left| \int_0^{t_2} \frac{|g(s, x_n(s))|}{(t_2 - s)^{1-\beta}} ds + \int_0^{t_1} \frac{|g(s, x_n(s))|}{(t_1 - s)^{1-\beta}} ds \right|$$

$$\le |q(t_2) - q(t_1)| + \frac{1}{\Gamma(\beta)} \left| \int_0^{t_2} \frac{h(s)}{(t_2 - s)^{1-\beta}} ds + \int_0^{t_1} \frac{h(s)}{(t_1 - s)^{1-\beta}} ds \right|$$

$$\le |q(t_2) - q(t_1)| + \frac{1}{\Gamma(\beta)} |v(t_2) - v(t_1)| \text{------}(3.10)$$

from the uniform continuity of the function $q(t), v(t)$ on $[0, T]$, we get $|Bx_n(t_2) - Bx_n(t_1)| \to 0$ as $t_1 \to t_2$.

Case II: If $t_1, t_2 \ge T$ then we have

$$|Bx_n(t_2) - Bx_n(t_1)| \le \left| q(t_2) + \frac{1}{\Gamma(\beta)} \int_0^{t_2} \frac{g(s, x_n(s))}{(t_2 - s)^{1-\beta}} ds - q(t_1) - \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{g(s, x_n(s))}{(t_1 - s)^{1-\beta}} ds \right|$$

$$\le |q(t_2) - q(t_1)| + \left| \frac{1}{\Gamma(\beta)} \int_0^{t_2} \frac{g(s, x_n(s))}{(t_2 - s)^{1-\beta}} ds - \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{g(s, x_n(s))}{(t_1 - s)^{1-\beta}} ds \right|$$

$$\le |q(t_2) - q(t_1)| + \left| \frac{1}{\Gamma(\beta)} \int_0^{t_2} \frac{g(s, x_n(s))}{(t_2 - s)^{1-\beta}} ds \right| + \left| \frac{1}{\Gamma(\beta)} \int_0^{t_1} \frac{g(s, x_n(s))}{(t_1 - s)^{1-\beta}} ds \right|$$

$$\le |q(t_2) - q(t_1)| + v(t_2) \frac{1}{\Gamma(\beta)} + v(t_1) \frac{1}{\Gamma(\beta)}$$

$$\le 0 + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \le \varepsilon \text{ as } t_1 \to t_2. \ (3.11)$$

Case III: if $t_1, t_2 \in R_+$, with $t_1 < T < t_2$, then we have

$$|Bx_n(t_2) - Bx_n(t_1)| \le |Bx_n(t_2) - Bx_n(T)| + |Bx_n(T) - Bx_n(t_1)| \qquad (3.12)$$

Note that if $t_1 \to t_2$, then $t_1 \to T$ and $T \to t_2$.

Therefore from the above obtained estimates, it follows that:

$|Bx_n(t_2) - Bx_n(T)| \to 0$; $|Bx_n(T) - Bx_n(t_1)| \to 0$ as $t_1 \to t_2$.

and so, $|Bx_n(t_2) - Bx_n(t_1)| \to 0$ as $t_1 \to t_2$. Thus, the estimate (3.12) holds for all $t_1, t_2 \in R_+$

This show $\{Bx_n\}$ is an equicontinuous sequence of functions in $B(B_r[0])$. Now an application of the Arzela-Ascoli theorem gives that $\{Bx_n\}$ has a convergent subsequence in $B(B_r[0])$ and consequently $B(B_r[0])$ is a relatively compact subset of X.This shows that B is compact operator on $B_r[0]$. As a results B is compact and continuous operator on $B_r[0]$ i.e. completely continuous operator on $B_r[0]$.

Step IV: Next we show that $AxBx \in B_r[0]$ for all $x \in B_r[0]$ is arbitrary, then

$$|Ax(t)Bx(t)| \le |Ax(t)||Bx(t)| \le |f(x(t))| \left(\left| q(t) + \frac{1}{\Gamma(\beta)} \int_0^t \frac{g(s, x(s))}{(t - s)^{1-\beta}} ds \right| \right)$$

$$\leq |f(x(t))| \left(|q(t)| + \frac{1}{\Gamma(\beta)} \int_0^t \frac{|g(s,x(s))|}{(t-s)^{1-\beta}} ds \right)$$
$$\leq F \left(|q(t)| + \frac{1}{\Gamma(\beta)} \int_0^t \frac{h(s)}{(t-s)^{1-\beta}} ds \right)$$
$$\leq F \left(|q(t)| + \frac{v(t)}{\Gamma(\beta)} \right)$$
$$\leq F(K_1 + K_2) = r \;\; for\, all\; t \in R_+$$

Taking the supremum over t, we obtain $\|AxBx\| \leq r$ for all $x \in B_r[0]$.

Hence hypothesis of Theorem 2.3 holds.

Also we have $M = \|B(B_r[0])\| = \sup\{\|Bx\| : x \in B_r[0]\}$

$$= sup \left\{ sup \left(|q(t)| + \frac{1}{\Gamma(\beta)} \int_0^t \frac{|g(s,x(s))|}{(t-s)^{1-\beta}} ds \right) : x \in B_r[0] \right\}$$
$$= sup \left\{ sup \left(|q(t)| + \frac{1}{\Gamma(\beta)} \int_0^t \frac{h(s)}{(t-s)^{1-\beta}} ds \right) : x \in B_r[0] \right\}$$
$$\leq sup|q(t)| + sup \frac{v(t)}{\Gamma(\beta)}$$
$$\leq (K_1 + K_2)$$

And therefore Mk = $L(K_1 + K_2) < 1$

Hence by Theorem 2.3 to conclude that (1.1) has a solution on R_+.

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