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SOME BILATERAL AND TRILATERALGENERATING RELATIONS INVOLVING A-FUNCTION

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ABSTRACT

The A-function of one variable plays an important role in the development and study of special functions. The usefulness of this function has inspired us to find some new generating relations. In this paper some new bilateral and trilateral generating relations have been established involving A-function of one variable and other hypergeometric functions.

1. INTRODUCTION:

The A-function of one variable is defined by Gautam [1] and we will represent here in the following manner:

$$A^{m,n}_{p,q}\left[x\,\Big|^{((a_p,\alpha_p))}_{((b_q,\beta_q))}\right] = \frac{1}{\quad}\int \theta(s)\, x^s\, ds \qquad (1.1)$$

where $i = \sqrt{(-1)}$ and

(i)

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(a_j + s\alpha_j)\prod_{j=1}^{n}\Gamma(1 - b_j - s\beta_j)}{\prod_{j=m+1}^{p}\Gamma(1 - a_j - s\alpha_j)\prod_{j=n+1}^{q}\Gamma(b_j + s\beta_j)} \qquad (1.2)$$

(ii) m, n, p and q are non-negative numbers in which $m \leq p$, $n \leq q$.

(iii) $x \neq 0$ and parameters a_j, α_j, b_k and β_k (j = 1 to p and k = 1 to q) are all complex.

The integral in the right hand side of is convergent if

(i) $x \neq 0, k = 0, h > 0, |\arg(ux)| < \pi h/2$

(ii) $x > 0, k = 0 = h, (\nu - \sigma\omega) < -1$

where

$$k = \text{Im}\left(\sum_1^p \alpha_j - \sum_1^q \beta_j\right) \quad (1.3)$$

$$h = \text{Re}\left(\sum_{j=1}^{m} \alpha_j - \sum_{j=m+1}^{p} \alpha_j + \sum_{j=1}^{n} \beta_j - \sum_{j=m+1}^{q} \beta_j\right) \quad (1.4)$$

$$u = \prod_1^p \alpha_j^{\alpha_j} \prod_1^q \beta_j^{\beta_j} \quad (1.5)$$

$$\nu = \text{Re}\left(\sum_1^p a_j - \sum_1^q b_j\right) - (p - q)/2, \quad (1.6)$$

$$w = \text{Re}\left(\sum_1^q \beta_j - \sum_1^p \alpha_j\right) \quad (1.7)$$

and $s = \sigma + it$ is on path L when $|t| \to \infty$.

In the present investigation we require the following formulae:

2. FORMULAE USED:

In the present investigation we require the following formulae:

From Shrivastava and Manocha [5],

$${}_1F_1[a; a; z] = e^z, \quad (2.1)$$

$|z| < 1$, $(1 - z)^{-a} = {}_1F_0[a; -; z]$, (2.2)

$e^z = {}_0F_0[-;-; z]$, (2.3)

$$(\alpha)_n = (\alpha, n) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \quad (2.4)$$

$$(1 - z)^{-a} = \sum_{n=0}^{\infty} (a)_n \frac{z^n}{n!}, \quad (2.5)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\mu)_n} P_n^{(\alpha-n,\beta-n)}(z)t^n = F_1\left[\lambda, -\alpha, -\beta; \mu; -(z+1)\frac{t}{2}, -(z-1)\frac{t}{2}\right], \quad (2.6)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n(\delta)_n}{(\alpha+1)_n(\beta+1)_n} P_n^{(\alpha,\beta)}(z)t^n = F_4\left[\lambda, \delta; \alpha+1, \beta+1; (z-1)\frac{t}{2}, (z+1)\frac{t}{2}\right]. \quad (2.7)$$

From Rainvile [2]:

$${}_2F_1\left[{}^{-n,\ a;}_{1+a+n;} -1\right] = \frac{(1+a)_n}{(1+a/2)_n}, \quad (2.8)$$

$$(\alpha)_{-n} = \frac{(-1)^n}{(1-\alpha,n)}, \quad (2.9)$$

$$(\alpha', p-q) = (\alpha', -q)(\alpha'-q, p) = \frac{(-1)^q(\alpha'-q,p)}{(1-\alpha',q)}, \quad (2.10)$$

$(\mu, p)\ (\mu + p, r + s) = (\mu, p + r + s)$, (2.11)

$(\lambda, p + q)\ (\lambda + p + q, r + s) = (\lambda, p + q + r + s)$

$= (\lambda, q)\ (\lambda + q, p + r + s)$, (2.12)

$(\mu, n)\ (\mu + n, p) = (\mu, n + p) = (\mu, p)\ (\mu + p, n)$. (2.13)

3.BILATERAL GENERATING RELATIONS:

In this section we establish the following bilateral Generating Relations:

$$\sum_{l=0}^{\infty} \frac{t^l}{l!} \quad {}_2F_1\left[{}^{-n,\ a;}_{1+a+n;} -1\right] A_{p,q+1}^{m,n+1}\left[x\Big|{}^{(a_j,\alpha_j)_{1,p}}_{(-a/2-n,0),(b_j,\beta_j)_{1,q}}\right]$$

$$= (1-t)^{-(a+1)} A^{m,n+1}_{p,q+1}\left[x\Big|^{(a_j,\alpha_j)_{1,p}}_{(-a/2,0),(b_j,\beta_j)_{1,q}}\right], \quad (3.1)$$

$$\sum_{l=0}^{\infty}\frac{t^l}{l!}\ {}_2F_1[^{-n,\ a;}_{1+a+n;}-1]A^{m+1,n}_{p+1,q}\left[x\Big|^{(1+a/2+n,0),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]$$

$$= (1-t)^{-(a+1)} A^{m+1,n}_{p+1,q}\left[x\Big|^{(1+a/2,0),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right], \quad (3.2)$$

$$\sum_{l=0}^{\infty}\frac{t^l}{l!}\ {}_2F_1[^{-n,\ a;}_{1-a-n;}-1]A^{m+1,n}_{p+1,q}\left[x\Big|^{(a+n,0),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]$$

$$= (1-t)^{-a/2} A^{m+1,n}_{p+1,q}\left[x\Big|^{(a,0),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right], \quad (3.3)$$

$$\sum_{l=0}^{\infty}\frac{t^l}{l!}\ {}_2F_1[^{-n,-a;}_{1-a-n;}-1]A^{m,n+1}_{p,q+1}\left[x\Big|^{(a_j,\alpha_j)_{1,p}}_{(1-a-n,0),(b_j,\beta_j)_{1,q}}\right]$$

$$= (1-t)^{-a/2} A^{m,n+1}_{p,q+1}\left[x\Big|^{(a_j,\alpha_j)_{1,p}}_{(1-a,0),(b_j,\beta_j)_{1,q}}\right]; \quad (3.4)$$

$|\arg(ux)| < \frac{1}{2}\pi h$, where h and u are given in (1.4) and (1.5) respectively.

Proof:

To prove (3.1), consider

$$\Delta = \sum_{l=0}^{\infty} \frac{t^l}{l!} \; {}_2F_1[{}^{-n,\ a;}_{1+a+n;} - 1] A^{m,n+1}_{p,q+1}\left[x\Big|^{(a_j,\alpha_j)_{1,p}}_{(-a/2-n,0),(b_j,\beta_j)_{1,q}}\right].$$

On expressing A-function in contour integral form as given in (1.1) and using (2.8), we get

$$\Delta = \sum_{l=0}^{\infty} \frac{t^l}{l!} \frac{(1+a)_n}{(1+a/2)_n} [\frac{1}{2\pi\omega}\int_L \theta(s)x^s\Gamma\{1-\left(-\frac{a}{2}-n\right)-0s\}\, ds].$$

In the view of (2.4) and (2.5), we arrive at R.H.S. of (3.1) as follows:

$$\Delta = \sum_{l=0}^{\infty} \frac{t^l}{l!} \frac{(1+a)_n}{(1+a/2)_n} [\frac{1}{2\pi\omega}\int_L \theta(s)x^s \left(1+\frac{a}{2}\right)_n \Gamma(1+a/2)\, ds]$$

$$= \frac{1}{2\pi\omega}\int_L \theta(s)x^s\Gamma\left(1+\frac{a}{2}\right)[\sum_{l=0}^{\infty}\frac{t^l}{l!}(1+a)_n]$$

$$= \frac{1}{2\pi\omega}\int_L \theta(s)x^s\Gamma\left(1+\frac{a}{2}\right)(1-t)^{-(a+1)}\, ds$$

$$= (1-t)^{-(a+1)} A^{m,n+1}_{p,q+1}\left[x\Big|^{(a_j,\alpha_j)_{1,p}}_{(-a/2,0),(b_j,\beta_j)_{1,q}}\right].$$

Proceeding on similar lines as above, the results (3.2) to (3.4) can be derived easily with the help of results given in section 2.

4. TRILATERAL GENERATING RELATIONS:

In this section we establish the following trilateral generating relations:

$$\sum_{n=0}^{\infty} H_2\,[\alpha',\beta',\gamma',\delta';\mu+n;x,y]P_n^{(\alpha-n,\beta-n)}(z)$$

$$.\,A_{p+1,q+1}^{m+1,n}\left[v\Big|_{(b_j,\beta_j)_{1,q},(\mu+n,0)}^{(\lambda+n,0),(a_j,\alpha_j)_{1,p}}\right]t^n$$

$$=\sum_{q=0}^{\infty}\frac{(\gamma',q)(\delta',q)}{(1-\alpha',q)(1,q)}(-y)^q A_{p+1,q+1}^{m+1,n}\left[v\Big|_{(b_j,\beta_j)_{1,q},(\mu,0)}^{(\lambda,0),(a_j,\alpha_j)_{1,p}}\right]$$

$$.F_S[\alpha'-q,\lambda,\lambda,\beta',-\alpha,-\beta;\mu,\mu,\mu;x,-(z+1)\tfrac{t}{2},-(z-1)\tfrac{t}{2}],$$

(4.1)

$|x| < r$, $|y| < s$, $(r + s) = 1$, $|\arg(uv)| < \frac{1}{2}\pi h$, where h and u are given in (1.4) and (1.5) respectively;

$$\sum_{n=0}^{\infty} G_1\,[\delta+n,\beta',\beta'';x,y]P_n^{(\alpha,\beta)}(z)$$

$$.A_{p+2,q+2}^{m+2,n}\left[v\Big|_{(b_j,\beta_j)_{1,q},(\alpha+1+n,0)\,(\beta+1+n,0)}^{(\gamma+n,0),(\delta+n,0),(a_j,\alpha_j)_{1,p}}\right]t^n$$

$$=\sum_{p=0}^{\infty}\frac{(\delta,p)(\beta'',p)}{(1-\beta',p)(1,p)}(-x)^p A_{p+2,q+2}^{m+2,n}\left[v\Big|_{(b_j,\beta_j)_{1,q},(\alpha+1,0)\,(\beta+1,0)}^{(\gamma,0),(\delta,0),(a_j,\alpha_j)_{1,p}}\right]$$

$$.\,F_E[\delta+p,\delta+p,\delta+p,\beta'-p,\gamma,\gamma;1-\beta''-p,\alpha+1,\beta+1;-y,(z-1)\tfrac{t}{2},(z+1)\tfrac{t}{2}],$$

(4.2)

|x| < r, |y| < s, (r + s) = 1, |x| < r, |y| < s, (r + s) = 1, |arg (uv)| < ½ πh, where h and u are given in (1.4) and (1.5) respectively;

$$\sum_{n=0}^{\infty} H_3\,[\alpha', \lambda + n; \mu + n; x, y] P_n^{(\alpha-n,\beta-n)}(z)$$

$$.A_{p+1,q+1}^{m+1,n}\left[v\Big|_{(b_j,\beta_j)_{1,q},(\mu+n,0)}^{(\lambda+n,0),(a_j,\alpha_j)_{1,p}}\right] t^n$$

$$= \sum_{p=0}^{\infty} \frac{(\alpha', 2p)}{(\mu, p)(1, p)} (x)^p A_{p+1,q+1}^{m+1,n}\left[v\Big|_{(b_j,\beta_j)_{1,q},(\mu,0)}^{(\lambda,0),(a_j,\alpha_j)_{1,p}}\right]$$

$$.F_N[\alpha' + 2p, -\alpha, -\beta, \lambda + r, \lambda, \lambda + r; \mu, \mu + q, \mu + q; y, -(z+1)\tfrac{t}{2}, -(z-1)\tfrac{t}{2}],$$

(4.3)

|x| < 1, |arg (uv)| < ½ πh, where h and u are given in (1.4) and (1.5) respectively;

$$\sum_{n=0}^{\infty} H_6\,[\alpha', \lambda + n; \gamma'; x, y] P_n^{(\alpha-n,\beta-n)}(z)$$

$$.A_{p+1,q+1}^{m,n+1}\left[v\Big|_{(1-\lambda-n,0),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}.(1-\mu-n,0)}\right] t^n$$

$$= \sum_{p=0}^{\infty} \frac{(\alpha', 2p)}{(1-\lambda, p)(1, p)} (-x)^p A_{p+1,q+1}^{m,n+1}\left[v\Big|_{(1-\lambda,0),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}.(1-\mu,0)}\right]$$

$$.F_G[\lambda - p, \lambda - p, \lambda - p, \gamma, -\alpha, -\beta; 1 - \alpha' - 2p, \mu, \mu; -y, -(z+1)\tfrac{t}{2}, -(z-1)\tfrac{t}{2}],$$

(4.4)

$|x| < r$, $|y| < s$, $rs^2 + s - 1$, $|\arg(uv)| < \frac{1}{2}\pi h$, where h and u are given in (1.4) and (1.5) respectively;

$$\sum_{n=0}^{\infty} H_7\,[\alpha', \gamma+n, \delta+n; \delta'; x, y] P_n^{(\alpha,\beta)}(z)$$

$$.\,A_{p+2,q+2}^{m+2,n}\left[v\Big|_{(b_j,\beta_j)_{1,q},(\alpha+1+n,0)\,(\beta+1+n,0)}^{(\gamma+n,0),(\delta+n,0),(a_j,\alpha_j)_{1,p}}\right] t^n$$

$$= \sum_{p=0}^{\infty} \frac{(\alpha', 2p)}{(\delta', p)(1, p)} (-x)^p A_{p+2,q+2}^{m+2,n}\left[v\Big|_{(b_j,\beta_j)_{1,q},(\alpha+1,0)\,(\beta+1,0)}^{(\gamma,0),(\delta,0),(a_j,\alpha_j)_{1,p}}\right]$$

$$.\,F_K[\gamma, \gamma+q, \gamma+q, \delta+r, \delta, \delta+r; 1-\alpha'-2p, \alpha+1, \beta+1; -y, (z-1)\tfrac{t}{2}, (z+1)\tfrac{t}{2}],$$

(4.5)

$|x| < r$, $|y| < s$, $4r = (s^{-1} - 1)^2$, $|\arg(uv)| < \frac{1}{2}\pi h$, where h and u are given in (1.4) and (1.5) respectively ,H_1 to $H_7$are given by Horn [3] and F_E, F_F, F_G, F_K, F_M, F_H, F_P, F_R, F_S and $F_T$are given by Saran [4],

Proof:

To prove (4.1), consider

$$\sum_{n=0}^{\infty} H_2\,[\alpha', \beta', \gamma', \delta'; \mu+n; x, y] P_n^{(\alpha-n,\beta-n)}(z)$$

$$.\,A_{p+1,q+1}^{m+1,n}\left[v\Big|_{(b_j,\beta_j)_{1,q},(\mu+n,0)}^{(\lambda+n,0),(a_j,\alpha_j)_{1,p}}\right] t^n.$$

Expressing H_2 in series form,by using (2.8) and A-function (1.1) and using (2.4), we get

$$\Delta= \sum_{n=0}^{\infty}\sum_{p,q=0}^{\infty} \frac{(\alpha',p-q)(\beta',p)(\gamma',q)(\delta',q)}{(\mu+n,p)(1,p)(1,q)} x^p y^q P_n^{(\alpha-n,\beta-n)}(z)$$

$$.[\tfrac{1}{2\pi\omega}\int_L \theta(s)u^s \tfrac{(\lambda,n)\Gamma(\lambda)}{(\mu,n)\Gamma(\mu)} ds]t^n.$$

Now interchange the order of summation and integration and on using (2.13), we get

$$\Delta = \tfrac{1}{2\pi\omega}\int_L \theta(s)\tfrac{\Gamma(\lambda)}{\Gamma(\mu)} u^s$$

$$.\sum_{p,q=0}^{\infty} \frac{(\alpha',p-q)(\beta',p)(\gamma',q)(\delta',q)}{(\mu,p)(1,p)(1,q)} x^p y^q$$

$$.\left[\sum_{n=0}^{\infty} \frac{(\lambda,n)}{(\mu+p,n)} P_n^{(\alpha-n,\beta-n)}(z)t^n\right] ds.$$

Again applying (3.2.6), we find that

$$\Delta = \frac{1}{2\pi\omega}\int_L \theta(s)u^s \frac{\Gamma(\lambda)}{\Gamma(\mu)} \sum_{p,q=0}^{\infty} \frac{(\alpha',p-q)(\beta',p)(\gamma',q)(\delta',q)}{(\mu,p)(1,p)(1,q)} x^p y^q$$

$$.F_1\left[\lambda,-\alpha,-\beta;\mu+p;-(z+1)\frac{t}{2},-(z-1)\frac{t}{2}\right] ds.$$

Further writing F_1 in series form, on using (2.2), we find that

$$\Delta = \frac{1}{2\pi\omega}\int_L \theta(s)u^s \frac{\Gamma(\lambda)}{\Gamma(\mu)} \sum_{p,q=0}^{\infty} \frac{(\alpha',p-q)(\beta',p)(\gamma',q)(\delta',q)}{(\mu,p)(1,p)(1,q)} x^p y^q$$

$$.\sum_{j,k=0}^{\infty} \frac{(\lambda,j+k)(-\alpha,j)(-\beta,k)}{(\mu+p,j+k)(1,j)(1,k)} [-(z+1)\frac{t}{2}]^j [-(z-1)\frac{t}{2}]^k ds.$$

Now using relation (2.10) and (2.11), we find that

$$\Delta = \frac{1}{2\pi\omega}\int_L \theta(s)u^s \frac{\Gamma(\lambda)}{\Gamma(\mu)}\sum_{q=0}^{\infty}\frac{(\gamma',q)(\delta',q)}{(1-\alpha',q)(1,q)}(-y)^q$$

$$\cdot\sum_{p,j,k=0}^{\infty}\frac{(\alpha'-q,p)(\lambda,j+k)(\beta',p)(-\alpha,j)(-\beta,k)}{(\mu,p+j+k)(1,p)(1,j)(1,k)}[-(z+1)\frac{t}{2}]^j[-(z-1)\frac{t}{2}]^k ds,$$

which provides (4.1). Proceeding on similar lines, (4.2) to (4.5) can be derived with the help of the formulae given in section 3.2.

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