CONHARMONIC CURVATURE TENSOR OF (ε)-PARA SASAKIAN 3- MANIFOLDS

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Abstract: *The present paper deals with the study of 3-dimensional (ε)-para Sasakian manifolds satisfying certain curvature conditions on the conharmonic curvature tensor $\overline{C}$. We characterize 3-dimensional (ε)-para Sasakian manifold satisfying ξ-conharmonically flat, $\overline{C}(X,Y)\cdot S=0$ and locally ϕ-recurrent condition, where S is Ricci tensor.*

Keywords - *(ε)-para Sasakian manifold, conharmonic curvature tensor, Einstein manifold.*

1. Introduction

In 1969, Takahashi [12] initiated the study of almost contact manifolds associated with an indefinite metrics. These indefinite almost contact metric manifolds are also called as (ε)-almost contact metric manifolds. The study of indefinite metric manifolds is of interest from the standpoint of physics and relativity. Indefinite metric manifolds have been studied by several authors. In 1993, Bejancu and Duggal [1] introduced the concept of (ε)-Sasakian manifolds. Some interesting properties of these manifolds were studied in [3, 4, 6, 14]. In 2009, De and Sarkar [2] introduced the concept of (ε)-Kenmotsu manifolds and showed that the existence of new structure on an indefinite metrics influences the curvatures.Tripathi and his co-authors [13] initiated the study of (ε)-almost paracontact metric manifolds, which is not necessarily Lorentzian. In particular, they studied (ε)-para Sasakian manifolds, with the structure vector field ξ is spacelike or timelike according as $\varepsilon=1$ or $\varepsilon=-1$. An (ε)-almost contact metric manifold is always odd dimensional but an (ε)-almost paracontact metric manifold could be even dimensional as well. Later, Perktas and his co-authors [7] studied (ε)-para Sasakian manifolds in dimension 3.

The present paper is organized as follows: After preliminaries, in Section 3, we study ξ-conharmonically flat (ε)-para Sasakian manifolds of dimension 3. Section 4 is devoted to study 3-dimensional (ε)-para Sasakian manifold satisfying the curvature condition $\overline{C}(X,Y)\cdot S=0$. Finally, in section 5, we study 3-dimensional locally conharmonically ϕ-recurrent (ε)-para Sasakian manifold.

2. Preliminaries

Let M be an n-dimensional almost paracontact manifold equipped with an almost paracontact structure (ϕ,ξ,η) consisting of a tensor field ϕ of type $(1,1)$, vector field ξ and a 1-form η satisfying

$$\phi^2=I-\eta\otimes\xi,\quad \eta(\xi)=1,\quad \phi\xi=0,\quad \eta\circ\phi=0. \tag{2.1}$$

Let g be a semi-Reimannian metric with index$(g)=\nu$ such that

$$g(\phi X,\phi Y)=g(X,Y)-\varepsilon\eta(X)\eta(Y), \tag{2.2}$$

where $\varepsilon = \pm 1$, then M is called an (ε)-almost paracontact metric manifold equipped with an (ε)-almost paracontact metric structure $(\phi, \xi, \eta, g, \varepsilon)$. In particular, if index$(g) = 1$, then an (ε)-almost paracontact metric manifold is said to be a Lorentzian almost paracontact manifold. In particular if the metric g is positive definite, then (ε)-almost paracontact metric manifold is the almost paracontact metric manifold [10]. The equation (2.2) is equivalent to

$$g(X, \phi Y) = g(\phi X, Y), \tag{2.3}$$

along with

$$g(X, \xi) = \varepsilon\eta(X). \tag{2.4}$$

From (2.1) and (2.4), it follows that

$$g(\xi, \xi) = \varepsilon. \tag{2.5}$$

An (ε)-almost paracontact metric structure is called an (ε)-para Sasakian structure if

$$(\nabla_X \phi)(Y) = g(\phi X, \phi Y)\xi - \varepsilon\eta(Y)\phi^2 X,\ X, Y \in TM, \tag{2.6}$$

where ∇ is the Levi-Civita connection with respect to g. A manifold endowed with (ε)-para Sasakian structure is called an (ε)-para Sasakian manifold.

For $\varepsilon = 1$ and g Riemannian, M is the usual para Sasakian manifold [9, 11]. For $\varepsilon = 1$, g Lorentzian and ξ replace by $-\xi$, M becomes a Lorentzian para Sasakian manifold [7].

For an (ε)-para Sasakian manifold, it is easy to prove that

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$R(\xi, X)Y = \eta(Y)X - \varepsilon\, g(X, Y)\xi, \tag{2.8}$$

$$R(\xi, X)\xi = X - \eta(X)\xi, \tag{2.9}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.10}$$

$$S(\xi, \xi) = -(n-1), \tag{2.11}$$

$$\nabla_X \xi = \varepsilon\,\phi X. \tag{2.12}$$

It is well known that in a 3-dimensional semi-Riemannian manifold the conformal curvature tensor C vanishes. Hence we get the following:

$$R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y - \frac{r}{2}\left(g(Y, Z)QX - g(X, Z)QY\right), \tag{2.13}$$

$$QX = \left(\frac{r}{2} + \varepsilon\right)X - \left(\frac{r}{2} + 3\varepsilon\right)\eta(X)\xi, \tag{2.14}$$

$$S(X, Y) = \left(\frac{r}{2} + \varepsilon\right)g(X, Y) - \left(\frac{r\varepsilon}{2} + 3\right)\eta(X)\eta(Y), \tag{2.15}$$

In view of (2.13), (2.14) and (2.15), we obtain the following

$$\begin{aligned} R(X, Y)Z = {} & \left(\frac{r}{2} + 2\varepsilon\right)(g(Y, Z)X - g(X, Z)Y) - \left(\frac{\varepsilon r}{2} + 3\right)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X) \\ & + \left(\frac{r}{2} + 3\varepsilon\right)(g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi). \end{aligned} \tag{2.16}$$

The conharmonic curvature tensor $\overline{C}$ of type $(1,3)$ on a semi-Riemannian manifold M of dimension 3 is defined by [5]

$$\overline{C}(X, Y)Z = R(X, Y)Z - [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]. \tag{2.17}$$

for all vector fields X, Y and Z on M^3.

By virtue of (2.14), (2.15) and (2.16), we get

$$\overline{C}(X,Y)Z = \frac{r}{2}[g(X,Z)Y - g(Y,Z)X]. \quad (2.18)$$

A (ε)-para Sasakian manifold is said to be η-Einstein manifold if the Ricci curvature tensor S is of the form

$$S(X,Y) = \alpha\, g(X,Z) + \beta\eta(X)\eta(Y),$$

where α and β are the constants. In particular, if $\beta = 0,$ then the manifold is called Einstein manifold.

3. ξ-conharmonically flat (ε)-para Sasakian 3-manifolds

Definition 3.1: *A 3-dimensional (ε)-para Sasakian manifold is said to be ξ-conharmonically flat if it satisfies the condition*

$$\overline{C}(X,Y)\xi = 0, \quad (3.1)$$

for all vector fields X, Y on M^3.

Taking $Z = \xi$ in (2.18) and then using (2.4), we get

$$\overline{C}(X,Y)\xi = \frac{\varepsilon r}{2}[\eta(X)Y - \eta(Y)X]. \quad (3.2)$$

Let us suppose that M^3 is ξ-conharmonically flat (ε)-para Sasakian 3-manifold. Then from (3.1) and (3.2), we get

$$\frac{\varepsilon r}{2}[\eta(X)Y - \eta(Y)X] = 0, \quad (3.3)$$

which implies that $\varepsilon r = 0$. But $\varepsilon \neq 0$ gives $r = 0$.

Conversely, if $r = 0$, then from (3.2), we get $\overline{C}(X,Y)\xi = 0.$ Hence we can state the following theorem:

Theorem 3.1: *A (ε)-para Sasakian 3-manifold is ξ-conharmonically flat if and only if the scalar curvature tensor r vanishes identically.*

4. (ε)-para Sasakian 3-manifold satisfying $\overline{C}(X,Y)\cdot S = 0$

In an (ε)-para Sasakian 3-manifold M^3, the condition $(\overline{C}(X,Y)\cdot S)(U,V) = 0$ is equivalent to

$$S(\overline{C}(X,Y)U,V) + S(U,\overline{C}(X,Y)V) = 0. \quad (4.1)$$

Taking $X = Y = \xi$ in (4.1), we get

$$S(\overline{C}(\xi,Y)\xi,V) + S(\xi,\overline{C}(\xi,Y)V) = 0. \quad (4.2)$$

From (2.18) and (2.4), we can easily get the following

$$\overline{C}(X,Y)\xi = \frac{\varepsilon r}{2}[\eta(X)Y - \eta(Y)X], \quad (4.3)$$

and

$$\overline{C}(\xi,Y)Z = \frac{r}{2}[\varepsilon\eta(Z)Y - g(Y,Z)\xi]. \quad (4.4)$$

Considering first term from (4.2) and using (4.3), we obtain

$$S(\overline{C}(\xi,Y)\xi,V) = \frac{\varepsilon r}{2}[S(Y,V) + 2\eta(Y)\eta(V)]. \quad (4.5)$$

And considering the second term of the equation (4.2) and then using (4.4), we get

$$S(\xi,\overline{C}(\xi,Y)V)=\frac{r}{2}[2g(Y,V)-2\varepsilon\eta(Y)\eta(V)]. \tag{4.6}$$

In view of (4.2), (4.5) and (4.6), we get the following relation

$$\varepsilon r[S(Y,V)+2\varepsilon g(Y,V)]=0, \tag{4.7}$$

which implies that $\varepsilon r=0$ or $S(Y,V)+2\varepsilon g(Y,V)=0$.

But $\varepsilon\neq 0$ gives $r=0$ or

$$S(Y,V)=-2\varepsilon g(Y,V). \tag{4.8}$$

Hence we can state the following theorem:

Theorem 4.2: *In an (ε)-para Sasakian 3-manifold M^3 satisfying $\overline{C}(X,Y)\cdot S=0$ **is** either an Einstein manifold or the scalar curvature tensor r vanishes identically.*

From Theorem 3.1 and Theorem 4.2, we state the following corollary:

Corollary 4.3: *A non-Einstein (ε)-para Sasakian 3-manifold M^3 satisfying $\overline{C}(X,Y)\cdot S=0$ is ξ-conharmonically flat.*

5. Locally conharmonically ϕ-recurrent (ε)-para Sasakian 3-manifold Definition 5.2: ***A 3-dimensional (ε)-para Sasakian manifold M^3 is said to be locally conharmonically ϕ-recurrent if and only if there exist a non-zero one form A, such that***

$$\phi^2((\nabla_W\overline{C})(X,Y)Z)=A(W)\overline{C}(X,Y)Z), \tag{5.1}$$

for all vector fields X, Y, Z and W are orthogonal to ξ.

Taking the covariant differentiation of (2.18) with respect to W, we obtain the following

$$(\nabla_W\overline{C})(X,Y)Z=\frac{1}{2}dr(W)[g(X,Z)Y-g(Y,Z)X]. \tag{5.2}$$

Now applying ϕ^2 on both sides of (5.2), we get

$$\phi^2((\nabla_W\overline{C})(X,Y)Z)=\frac{1}{2}dr(W)[g(X,Z)\phi^2Y-g(Y,Z)\phi^2X]. \tag{5.3}$$

By virtue of (5.1) and (5.3), we have

$$A(W)\overline{C}(X,Y)Z=\frac{1}{2}dr(W)[g(X,Z)\phi^2Y-g(Y,Z)\phi^2X], \tag{5.4}$$

which yields by (2.1) that

$$A(W)\overline{C}(X,Y)Z=\frac{1}{2}dr(W)[g(X,Z)Y-g(Y,Z)X-g(X,Z)\eta(Y)\xi+g(Y,Z)(X)\xi]. \tag{5.5}$$

Noting that, we may assume that all the vector fields X, Y, Z and W are orthogonal to ξ, then we get

$$A(W)\overline{C}(X,Y)Z=\frac{1}{2}dr(W)[g(X,Z)Y-g(Y,Z)X]. \tag{5.6}$$

From above equation it follows that

$$\overline{C}(X,Y)Z=\frac{dr(W)}{2A(W)}[g(X,Z)Y-g(Y,Z)X]. \tag{5.7}$$

Putting $W = e_i$ in (5.7), where $e_i = 1, 2, 3$ is an orthonormal basis of the tangent space at any point of the manifold and taking the summation over $1 \le i \le 3$, we obtain

$$\overline{C}(X,Y)Z = \frac{dr(e_i)}{2A(e_i)}[g(X,Z)Y - g(Y,Z)X]. \tag{5.8}$$

In view of (2.16) and (2.18), we get

$$\begin{aligned}\overline{C}(X,Y)Z &= R(X,Y)Z + (r+2\varepsilon)(g(X,Z)Y - g(Y,Z)X) \\ &\quad -\left(\frac{\varepsilon r}{2}+3\right)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X) \\ &\quad -\left(\frac{r}{2}+3\varepsilon\right)(g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi).\end{aligned} \tag{5.9}$$

By virtue of (5.8) and (5.9), we have

$$\begin{aligned}R(X,Y)Z &= \left[\frac{dr(e_i)}{2A(e_i)} - (r+2\varepsilon)\right](g(X,Z)Y - g(Y,Z)X) \\ &\quad +\left(\frac{\varepsilon r}{2}+3\right)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X) \\ &\quad +\left(\frac{r}{2}+3\varepsilon\right)(g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi).\end{aligned} \tag{5.10}$$

Taking X, Y and W are orthogonal to ξ, we obtain from (5.10) that

$$R(X,Y)Z = \left[\frac{dr(e_i)}{2A(e_i)} - (r+2\varepsilon)\right](g(X,Z)Y - g(Y,Z)X). \tag{5.11}$$

Equivalently,

$$R(X,Y)Z = \lambda(g(X,Z)Y - g(Y,Z)X), \tag{5.12}$$

where $\lambda = \frac{dr(e_i)}{2A(e_i)} - (r+2\varepsilon)$ is a scalar. A 1-form A is non-zero, and then by Schur's theorem λ be a constant on the manifold. Therefore, M^3 is of constant curvature λ.

Hence, we can state the following theorem:

Theorem 5.1: *A locally conharmonically ϕ-recurrent (ε)-para Sasakian 3-manifold M^3 is of constant curvature.*

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