Some Classes of Set Graceful Graphs

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Abstract

The concept of Set Graceful and Set Sequential Graphs were introduced by B D Acharya in 1983. Let $G = (V, E)$ be a graph, X be a non empty set and 2^X denote the set of all subsets of X .A set indexer of G is an injective set valued function $f : V(G) \to 2^X$ such that the function $f^{\oplus} : E(G) \to 2^X - \{\emptyset\}$ defined by

$f^{\oplus}(UV) = f(U) \oplus f(V)$ for every $U, V \in E(G)$ is also injective, where $\oplus$ denotes the symmetric difference of sets.

A graph $G = (V, E)$ is said to be set graceful if there exist a nonempty set X and a set indexer $f : V(G) \to 2^X$ such that

$f^{\oplus}(E(G)) = 2^X - \{\emptyset\}$,such an indexer being called set graceful labeling of G. In this paper we identified necessary conditions for some binary operations on certain classes of Set Graceful graphs to be Set Graceful.

Keywords: Set indexer, Set Graceful Graphs, Biset Graceful Graphs.

1 Introduction

Motivated from number valuations of graphs, B.D.Acharya defined Set indexer,set graceful and set sequential graphs in 1983.Subsequently many researchers have worked with set graceful graph and found some classes of set graceful graph.Many problems in this area are to be solved.

In this paper we have studied some binary operations on certain classes of set Graceful graphs. Necessary conditions for certain graphs to be set graceful and biset graceful have been studied.

All the notations and terminologies used in this paper are from Harary[5]. All the graphs in this paper are simple and finite.

2 Preliminaries

Definition 2.1. [1]

Let $G = (V, E)$ be a graph, X be a non empty set and 2^X denote the set of all subsets of X.A set indexer of G is an injective set valued function

f: $V(G) \rightarrow 2^X$ such that the function $f^{\oplus} : E(G) \rightarrow 2^X - \{\emptyset\}$ defined by

$f^{\oplus}(UV) = f(U) \oplus f(V)$ for every $U, V \in E(G)$ is also injective, where $\oplus$ denotes the symmetric difference of sets.

Theorem 2.2. [1] Every graph has a set indexer.

Definition 2.3. [1]

A graph $G = (V, E)$ is said to be set graceful if there exist a nonempty set X and a set indexer **f**: $V(G) \rightarrow 2^X$ such that $f^{\oplus}(E(G)) = 2^X - \{\emptyset\}$,such an indexer being called set graceful labeling of G.

The following theorem gives a straight forward necessary condition for a graph G to be set graceful.

Theorem 2.4. [1]

If $G = (p, q)$ be a set graceful graph, then $q = 2^m - 1$

and $p \leq q + 1$.

The smallest such positive integer m is called the set indexing number.

Definition 2.5. [5]

Let G_1 and G_2 be two graphs with disjoint vertex sets V_1 and V_2 then, their Join $G_1 + G_2$ consists of $G_1 \cup G_2$ and all the edges joining V_1 with V_2.

Definition 2.6. [5]

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs. Then their product

$G_1 \times G_2$ is defined as follows, two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$ are adjacent in $G_1 \times G_2$ if $u_1 = v_1$ and u_2 adjacent to v_2 or $u_2 = v_2$ and u_1 adjacent to v_1.

Definition 2.7. [6]

The corona of two graphs $G_1 = (p_1, q_1)$ and $G_2 = (p_2, q_2)$ is the graph

$G_1 \odot G_2$ formed from one copy of G_1 and $/V(G_1)/$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2.

Theorem 2.8. [6]

The complete graph K_n is set graceful if and only if $n \in \{1, 2, 3, 6\}$.

Definition 2.9. [5]

The line graph of a graph G, denoted by L(G), has E(G) as its vertex set with two vertices of L(G) are adjacent whenever the corresponding edges of G are adjacent.

3 Somebinary operations on set gracefulgraphs.

Theorem 3.1.

A necessary condition for the cross product $G_1 \times G_2$ of two set graceful graphs $G_1 = (p_1, q_1)$ and $G_2 = (p_2, q_2)$ to be set graceful is that p_1 and p_2 are of opposite parities.

Proof: Let G_i has p_i vertices and q_i edges for i = 1, 2. Then $q_i = 2^{n_i} - 1$ where n_i are the set indexing number for each G_i. If $G_1 \times G_2$ is set graceful then

$$p_1(2^{n_2} - 1) + p_2(2^{n_1} - 1) = 2^k - 1 \text{ for some } k \in N.$$

Then $p_1 2^{n_2} + p_2 2^{n_1} = 2^k - 1 + p_1 + p_2$ .Hence $p_1 + p_2$ must be odd.

Therefore either p_1 is odd and p_2 is even or p_2 is odd and p_1 is even.

Hence the theorem.

Remark 3.2. Converse of the above result is not true as $K_3 \times K_2$ is not set graceful.

Theorem 3.3.

If T is a Set graceful tree with n vertices, then $K_1 + T$ is Set graceful.

Proof: Let $G = K_1 + T$. Since T is Set Graceful, T has 2^n vertices and $2^n - 1$ edges for some $n \in N$ .Take $V(T) = \{v_1, v_2, \ldots\ldots v_{2^n}\}$ and $V(K_1) = \{u\}$ so that $V(G) = \{u, v_1, v_2 \ldots . v_{2^n}\}$. Then $/V(G)/ = 2^n + 1$ and

$/E(G)/ = 2^{n+1} - 1$. Define $X_n = \{1, 2, 3, \ldots .n\}$. Define the labeling

$f: V(G) \rightarrow 2^{X_{n+1}}$ as follows.Label u byX_{n+1} and v_i by subsets of X_n,

for each $i = 1, 2, \ldots .2^n$ in such a way that T is Set graceful.Then the labelling

of edges of G are injective and

$f^{\oplus}(G) = 2^{X_{n+1}} - \{\varnothing\}$.Therefore G is Set graceful. □

Theorem 3.4.

A necessary condition for the corona $G_1 \odot G_2$ of two set graceful graphs $G_1 = (p_1, q_1)$ and $G_2 = (p_2, q_2)$ to be set graceful is that

$p_1 \equiv 0 (\text{mod} 2)$ or $p_2 \equiv 1 (\text{mod} 2)$ or both.

Proof: Let G_i has p_i vertices and q_i edges for i = 1, 2. Then $q_i = 2^{n_i} - 1$ where n_i are the set indexing number for each G_i . Then $G_1 \odot G_2$ has $p_1(1 + p_2)$ vertices and $q_1 + p_1(p_2 + q_2)$ edges.

Then if $G_1 \cup G_2$ is set graceful we have $q_1 + p_1(p_2 + q_2) = 2^k - 1$ for some $k \in N$.

Therefore $2^{n_1} - 1 + p_1 p_2 + p_1(2^{n_2} - 1) = 2^k - 1$.

Hence $2^{n_1} + p_1 2^{n_2} - 2^k = p_1 - p_1 p_2$.

That is 2 divides $p_1(1 - p_2)$.

Hence $p_1 \equiv 0(\text{mod}2)$ or $p_2 \equiv 1(\text{mod}2)$ or both.

Definition 3.5. [8]

The Boolean lattice BL_n is the graph whose vertex set is the set of all subsets of {1, 2, 3....., n} where two subsets X and Y are adjacent if there symmetric difference has precisely one element.

Obsesrvation 3.6. No Boolean lattice is set graceful as labeling of edges is not injective.

Definition 3.7. [6]

An n−sun is a graph that consists of a cycle C_n and an edge terminating in vertex of degree one attached to each vertex of C_n.

Observation 3.8. The n−sun is not set graceful for every n.

4 Biset Graceful Graphs

Definition 4.1. [6]

A graph G is called biset graceful if G and its line graph L(G) are set graceful.

Theorem 4.2.

If G = (p, q) is a biset graceful graph, then sum of squares of degrees of vertex is divisible by 4, when $p \geq 3$.

Proof: Suppose G = (p, q) be a biset graceful graph. Then $q = 2^m - 1$ for some

$m \in N$. Also $L(G) = (-q, -q + \sum d_i^2/2)$.Then $-q + \sum d_i^2/2 = 2^n - 1$ for some $n \in N$.

But then $-2^m+1+\sum d_i^2/2=2^n-1$.Therefore,$\sum d_i^2=4(2^{m-1}+2^{n-1}-1)$

Hence the result. □

Remark 4.3. When $p = 2, \sum_i di^2$ is divisible by 2.

Remark 4.4. P_2 is the only biset graceful path. Since no path $P_n, n \geq 3$ is set graceful and both P_2 and $L(P_2) = K_1$ are set graceful.

Problem 4.5. $K_3 \times K_6$ is set graceful

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