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SOME NEW IDENTITIES INVOLVING I-FUNCTION OF TWO VARIABLES

By

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ABSTRACT

Looking into the requirement and importance of various properties of identity in various field, in this paper we established some new identities involving I-function of two variables.

1. INTRODUCTION:

The I–function of two variables introduced by Sharma & Mishra [2], will be defined and represented as follows:

$$I\left[{x \atop y}\right] = I^{0,n:m_1,n_1:m_2,n_2}_{p_i,q_i:r:p_{i'},q_{i'}:r':p_{i''},q_{i''}:r''}\left[{x \atop y}\Bigg|{[(a_j:\alpha_j,A_j)_{1,n}],[(a_{ji}:\alpha_{ji},A_{ji})_{n+1,p_i}] \atop [(b_{ji}:\beta_{ji},B_{ji})_{1,q_i}]}\right.$$
$$\left.{:[(c_j;\gamma_j)_{1,n_1}],[(c_{ji'};\gamma_{ji'})_{n_1+1,p_{i'}}];[(e_j;E_j)_{1,n_2}],[(e_{ji''};E_{ji''})_{n_2+1,p_{i''}}] \atop :[(d_j;\delta_j)_{1,m_1}],[(d_{ji'};\delta\gamma_{ji'})_{m_1+1,q_{i'}}];[(f_j;F_j)_{1,m_2}],[(f_{ji''};F_{ji''})_{m_2+1,q_{i''}}]}\right]$$
$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)\,x^{\xi}y^{\eta}\,d\xi d\eta, \qquad (1)$$

where

$$\phi_1(\xi,\eta) = \frac{\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)}{\sum_{i=1}^{r}[\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi-A_{ji}\eta)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi+B_{ji}\eta)]},$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_1}\Gamma(d_j-\delta_j\xi)\prod_{j=1}^{n_1}\Gamma(1-c_j+\gamma_j\xi)}{\sum_{i'=1}^{r'}[\prod_{j=m_1+1}^{q_{i'}}\Gamma(1-d_{ji'}+\delta_{ji'}\xi)\prod_{j=n_1+1}^{p_{i'}}\Gamma(c_{ji'}-\gamma_{ji'}\xi)]},$$

$$\theta_3(\eta) = \frac{\prod_{j=1}^{m_2}\Gamma(f_j-F_j\eta)\prod_{j=1}^{n_2}\Gamma(1-e_j+E_j\eta)}{\sum_{i''=1}^{r''}[\prod_{j=m_2+1}^{q_{i''}}\Gamma(1-f_{ji''}+F_{ji''}\eta)\prod_{j=n_2+1}^{p_{i''}}\Gamma(e_{ji''}-E_{ji''}\eta)]},$$

x and y are not equal to zero, and an empty product is interpreted as unity p_i, $p_{i'}$, $p_{i''}$, q_i, $q_{i'}$, $q_{i''}$, n, n_1, n_2, n_j and m_k are non negative integers such that $p_i \geq n \geq 0$, $p_{i'} \geq n_1 \geq 0$, $p_{i''}$

$\geq n_2 \geq 0$, $q_i > 0$, $q_{i'} \geq 0$, $q_{i''} \geq 0$, ($i = 1, \ldots, r$; $i' = 1, \ldots, r'$; $i'' = 1, \ldots, r''$; $k = 1, 2$) also all the A's, α's, B's, β's, γ's, δ's, E's and F's are assumed to be positive quantities for standardization purpose; the definition of I-function of two variables given above will however, have a meaning even if some of these quantities are zero. The contour L_1 is in the ξ–plane and runs from $-\omega\infty$ to $+\omega\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(d_j-\delta_j\xi)$ ($j = 1, \ldots, m_1$) lie to the right, and the poles of $\Gamma(1 - c_j + \gamma_j\xi)$ ($j = 1, \ldots, n_1$), $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ ($j = 1, \ldots, n$) to the left of the contour.

The contour L_2 is in the η–plane and runs from $-\omega\infty$ to $+\omega\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(f_j - F_j\eta)$ ($j=1,\ldots, n_2$) lie to the right, and the poles of $\Gamma(1 - e_j + E_j\eta)$ ($j = 1, \ldots, m_2$), $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ ($j = 1, \ldots, n$) to the left of the contour. Also

$$R' = \sum_{j=1}^{p_i} \alpha_{ji} + \sum_{j=1}^{p_{i'}} \gamma_{ji'} - \sum_{j=1}^{q_i} \beta_{ji} - \sum_{j=1}^{q_{i'}} \delta_{ji'} < 0,$$

$$S' = \sum_{j=1}^{p_i} A_{ji} + \sum_{j=1}^{p_{i''}} E_{ji''} - \sum_{j=1}^{q_i} B_{ji} - \sum_{j=1}^{q_{i''}} F\delta_{ji'} < 0,$$

$$U = \sum_{j=n+1}^{p_i} \alpha_{ji} - \sum_{j=1}^{q_i} \beta_{ji} + \sum_{j=1}^{m_1} \delta_j - \sum_{j=m_1+1}^{q_{i'}} \delta_{ji'} + \sum_{j=1}^{n_1} \gamma_j - \sum_{j=n_1+1}^{p_{i'}} \gamma_{ji'} > 0, \tag{2}$$

$$V = -\sum_{j=n+1}^{p_i} A_{ji} - \sum_{j=1}^{q_i} B_{ji} - \sum_{j=1}^{m_2} F_j - \sum_{j=m_{21}+1}^{q_{i''}} F_{ji''} + \sum_{j=1}^{n_2} E_j - \sum_{j=n_2+1}^{p_{i''}} E_{ji''} > 0, \tag{3}$$

and $|\arg x| < \frac{1}{2} U\pi$, $|\arg y| < \frac{1}{2} V\pi$.

In the present investigation we require the following formula:

From Rainvile [1]:

$$\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} = \frac{\sin \pi\alpha}{\pi} = \frac{e^{i\pi\alpha} - e^{-i\pi\alpha}}{2\pi i}, \tag{4}$$

$$\Gamma\left(\frac{1}{2} - z\right)\Gamma\left(\frac{1}{2} + z\right) = \pi \sec\pi z, \tag{5}$$

$$\Gamma(z)\Gamma(1 - z) = \pi \operatorname{cosec}\pi z, \tag{6}$$

$$\Gamma(\alpha - r)\Gamma(1 - \alpha + r) = (-1)^r\Gamma(1 - \alpha)\Gamma(\alpha). \tag{7}$$

2. IDENTITIES INVOLVING I-FUNCTION OF TWO VARIABLES:

In this section, we will establish following identities involving I-function of two variables:

$$I^{0,\ n\ :m_1,n_1\ :m_2,n_2}_{p_i,q_i:r\ :p_i'+1,q_i'+1:r'\ :p_i'',q_i'':r''}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\Big|\begin{matrix} \ldots\ldots,:\ldots\ldots,(\sigma,\mu):\ldots\ldots,\ldots\ldots \\ \ldots\ldots,:\ldots\ldots,(\sigma,\mu):\ldots\ldots,\ldots\ldots \end{matrix}\right]$$

$$= (2\pi i)^{-1}\{e^{i\pi\sigma}\, I^{0,\ n\ :m_1,n_1\ :m_2,n_2}_{p_i,q_i:r\ :p_i',q_i':r'\ :p_i'',q_i'':r''}\left[\begin{matrix} z_1 e^{-i\pi\mu} \\ z_2 \end{matrix}\Big|\begin{matrix} \ldots\ldots,:\ldots\ldots:\ldots\ldots,\ldots\ldots \\ \ldots\ldots,:\ldots\ldots:\ldots\ldots,\ldots\ldots \end{matrix}\right]$$

$$-e^{-i\pi\sigma}\, I^{0,\ n\ :m_1,n_1\ :m_2,n_2}_{p_i,q_i:r\ :p_i',q_i':r'\ :p_i'',q_i'':r''}\left[\begin{matrix} z_1 e^{i\pi\mu} \\ z_2 \end{matrix}\Big|\begin{matrix} \ldots\ldots,:\ldots\ldots:\ldots\ldots,\ldots\ldots \\ \ldots\ldots,:\ldots\ldots:\ldots\ldots,\ldots\ldots \end{matrix}\right]\}, \qquad (8)$$

$|\arg z_1| < ½\, U\pi$, $|\arg z_2| < ½\, V\pi$, where U and V is given in (2) and (3) respectively;

$$e^{i\pi\sigma}\, I^{0,\ n\ :m_1,n_1\ :m_2,n_2}_{p_i,q_i:r\ :p_i',q_i':r'\ :p_i'',q_i'':r''}\left[\begin{matrix} z_1 e^{i\pi\mu} \\ z_2 \end{matrix}\Big|\begin{matrix} \ldots\ldots,:\ldots\ldots:\ldots\ldots,\ldots\ldots \\ \ldots\ldots,:\ldots\ldots:\ldots\ldots,\ldots\ldots \end{matrix}\right]$$

$$= \pi\{I^{0,\ n\ :m_1,n_1\ :m_2,n_2}_{p_i,q_i:r\ :p_i'+1,q_i'+1:r'\ :p_i'',q_i'':r''}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\Big|\begin{matrix} \ldots\ldots,:\ldots\ldots,\left(\tfrac{1}{2}-\sigma,\mu\right):\ldots\ldots,\ldots\ldots \\ \ldots\ldots,:\ldots\ldots,\left(\tfrac{1}{2}-\sigma,\mu\right):\ldots\ldots,\ldots\ldots \end{matrix}\right]$$

$$+\, i\, I^{0,\ n\ :m_1,n_1\ :m_2,n_2}_{p_i,q_i:r\ :p_i'+1,q_i'+1:r'\ :p_i'',q_i'':r''}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\Big|\begin{matrix} \ldots\ldots,:\ldots\ldots,(1-\sigma,\mu):\ldots\ldots,\ldots\ldots \\ \ldots\ldots,:\ldots\ldots,(1-\sigma,\mu):\ldots\ldots,\ldots\ldots \end{matrix}\right]\}, \qquad (9)$$

$|\arg z_1| < ½\, U\pi$, $|\arg z_2| < ½\, V\pi$, where U and V is given in (2) and (3) respectively;

$$I^{0,\ n\ :m_1,n_1\ :m_2,n_2}_{p_i,q_i:r\ :p_i'+1,q_i'+1:r'\ :p_i'',q_i'':r''}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\Big|\begin{matrix} \ldots\ldots,:\ldots\ldots,(\sigma+r,\mu):\ldots\ldots,\ldots\ldots \\ \ldots\ldots,:\ldots\ldots,(\sigma+r,\mu):\ldots\ldots,\ldots\ldots \end{matrix}\right]$$

$$= (-1)^r\, I^{0,\ n\ :m_1,n_1\ :m_2,n_2}_{p_i,q_i:r\ :p_i'+1,q_i'+1:r'\ :p_i'',q_i'':r''}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\Big|\begin{matrix} \ldots\ldots,:\ldots\ldots,(\sigma,\mu):\ldots\ldots,\ldots\ldots \\ \ldots\ldots,:\ldots\ldots,(\sigma,\mu):\ldots\ldots,\ldots\ldots \end{matrix}\right], \qquad (10)$$

$|\arg z_1| < ½\, U\pi$, $|\arg z_2| < ½\, V\pi$, where U and V is given in (2) and (3) respectively.

Proof:

To establish (8), replace I-function of two variables by its equivalent contour integral as given in (1), after using (4) and finally interpreting by (1), we obtain (8).

Proceeding on the same lines the results (9) and (10) can be easily established.

3. PARTICULAR CASES:

I. On specializing the parameters in main result, we get following identities in terms of I-function of one variable:

$$I^{m,n}_{p_i+1,q_i+1:r}\left[z\Big|\begin{matrix} \ldots\ldots,\ldots\ldots,(\sigma,\mu) \\ \ldots\ldots,\ldots\ldots,(\sigma,\mu) \end{matrix}\right]$$

$$= (2\pi i)^{-1}\{e^{i\pi\sigma}\, I^{m,n}_{p_i,q_i:r}\left[ze^{-i\pi\mu}\Big|\begin{matrix} \ldots\ldots,\ldots\ldots \\ \ldots\ldots,\ldots\ldots \end{matrix}\right] - e^{-i\pi\sigma}\, I^{m,n}_{p_i,q_i:r}\left[ze^{i\pi\mu}\Big|\begin{matrix} \ldots\ldots,\ldots\ldots \\ \ldots\ldots,\ldots\ldots \end{matrix}\right]\}, \qquad (11)$$

provided that $|\arg z| < ½\, B\pi$, where B is given by $B = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p_i}\alpha_{ji} + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q_i}\beta_{ji} > 0$;

$$e^{i\pi\sigma}\, I^{m,n}_{p_i,q_i:r}\left[ze^{i\pi\mu}\Big|\begin{matrix} \ldots\ldots,\ldots\ldots \\ \ldots\ldots,\ldots\ldots \end{matrix}\right]$$

$$= \pi\{I^{m,n}_{p_i+1,q_i+1:r}[z|^{\ldots\ldots,\ldots\ldots,(\frac{1}{2}-\sigma,\mu)}_{\ldots\ldots,\ldots\ldots,(\frac{1}{2}-\sigma,\mu)}] + iI^{m,n}_{p_i+1,q_i+1:r}[z|^{\ldots\ldots,\ldots\ldots,(1-\sigma,\mu)}_{\ldots\ldots,\ldots\ldots,(1-\sigma,\mu)}]\}, \quad (12)$$

provided that $|\arg z| < \frac{1}{2} B\pi$, where B is given by $B = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p_i} \alpha_{ji} + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q_i} \beta_{ji} > 0$;

$$I^{m,n}_{p_i+1,q_i+1:r}[z|^{\ldots\ldots,\ldots\ldots,(\sigma+r,\mu)}_{\ldots\ldots,\ldots\ldots,(\sigma+r,\mu)}]$$

$$= (-1)^r I^{m,n}_{p_i+1,q_i+1:r}[z|^{\ldots\ldots,\ldots\ldots,(\sigma,\mu)}_{\ldots\ldots,\ldots\ldots,(\sigma,\mu)}], \quad (13)$$

provided that $|\arg z| < \frac{1}{2} B\pi$, where B is given by $B = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p_i} \alpha_{ji} + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q_i} \beta_{ji} > 0$;

II. On choosing r = 1 in the integrals (11) to (13), we get following identities in terms of H-function of one variable:

$$H^{m,n}_{p+1,q+1}[z|^{(a_j,\alpha_j)_{1,p},(\sigma,\mu)}_{(b_j,\beta_j)_{1,q},(\sigma,\mu)}]$$

$$= (2\pi i)^{-1}\{e^{i\pi\sigma} H^{m,n}_{p,q}[ze^{-i\pi\mu}|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}] - e^{-i\pi\sigma} H^{m,n}_{p,q}[ze^{i\pi\mu}|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}]\}, \quad (14)$$

provided that $|\arg z| < \frac{1}{2} A\pi$, where A is given by $A = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j > 0$;

$$e^{i\pi\sigma} H^{m,n}_{p,q}[ze^{i\pi\mu}|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}]$$
$$= \pi\{H^{m,n}_{p+1,q+1}[z|^{(a_j,\alpha_j)_{1,p},(\frac{1}{2}-\sigma,\mu)}_{(b_j,\beta_j)_{1,q},(\frac{1}{2}-\sigma,\mu)}] + iH^{m,n}_{p+1,q+1}[z|^{(a_j,\alpha_j)_{1,p},(1-\sigma,\mu)}_{(b_j,\beta_j)_{1,q},(1-\sigma,\mu)}]\}; \quad (15)$$

provided that $|\arg z| < \frac{1}{2} A\pi$, where A is given by $A = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j > 0$;

$$H^{m,n}_{p+1,q+1}[z|^{(a_j,\alpha_j)_{1,p},(\sigma+r,\mu)}_{(b_j,\beta_j)_{1,q},(\sigma+r,\mu)}] = (-1)^r H^{m,n}_{p+1,q+1}[z|^{(a_j,\alpha_j)_{1,p},(\sigma,\mu)}_{(b_j,\beta_j)_{1,q},(\sigma,\mu)}], \quad (16)$$

provided that $|\arg z| < \frac{1}{2} A\pi$, where A is given by $A = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j > 0$;

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