ON K-ECCENTRIC INDICES AND K HYPER- ECCENTRIC INDICES OF GRAPHS.

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ABSTRACT: *We define First and second K-Eccentric indices as,* $B_1E(G) = \sum_{ue}\left[e_G(u) + e_{L(G)}(e)\right]$ *and* $B_2E(G) = \sum_{ue}\left[e_G(u)e_{L(G)}(e)\right]$, *and the First and second K Hyper-Eccentric indices as* $HB_1E(G) = \sum_{ue}\left[e_G(u) + e_{L(G)}(e)\right]^2$ *and* $HB_2E(G) = \sum_{ue}\left[e_G(u)\, e_{L(G)}(e)\right]^2$.

KEYWORDS: Chemical graph, Eccentricity index, First and second K Eccentric indices, First and second K Hyper Eccentric indices, and Topological index. .

INTRODUCTION:

In the field of chemical graph the molecular topology and mathematical chemistry, a topological index known as connectivity index is a type of a molecular descriptor that is calculated based on the molecular graph of a chemical compound. Topological indices are the numerical parameters of a graph which characterize its topology and are usually graph invariant. Topological indices are used for example in the development of quantitative structure activity relationship (QSAR'S) in which the biological activity or other properties of molecules are correlated with their chemical structure.

Let $G = (V, E)$ be a graph with $|V| = n \; and \; |E| = m$. The eccentricity $e_G(v)$ of a vertex v is the distance of any vertex farthest from v. Let $e = uv \in E(G)$. Let $e_{L(G)}(e)$ denote the eccentricity of an edge e in L(G), where L(G) is the line graph of G. The vertices and edges of a graph are called the elements of G. The line graph of an undirected graph G is another graph L(G) that represents the adjacencies between edges of G. A Line graph of a simple graph is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge iff the corresponding edges of G have a vertex in common. The concept of Zagreb eccentricity (E_1 and E_2) indices was introduced by Vukicevic and Gravoc in the chemical graph theory very recently $[1-4]$. The first Zagreb eccentricity (E_1) and the second Zagreb eccentricity (E_2) indices of a graph G are defined as $E_1(G) = \sum_{v_i \in V(G)} {e_i}^2$ and $E_2(G) = \sum_{v_i v_j \in E(G)} e_i e_j$ where E(G) is the edge set and e_i is the eccentricity of the vertex v_i in G. The multiplicative variant of Zagreb indices was introduced by Todeschini et. al [5]. They are defined as $\prod_1(G) = \prod_{v \epsilon V(G)} (deg_G(v))^2$ and $\prod_2(G) = (deg_G(u))(deg_G(v))$. Also we recently defined Harmonic Eccentric index and it is defined as $HEI(G) = \sum_{uv \epsilon E\,(G)} = \frac{2}{e_u + e_v}$, where e_u the is eccentricity of the vertex u in G.

In [6], Kulli introduced the first and second K Banhatti indices to take account of the contributions of pairs of incident elements. and it was defined as

$$B_1(G) = \sum_{ue}[d_G(u) + d_G(e)]. \qquad (1)$$

$$B_2(G) = \sum_{ue}[d_G(u)d_G(e)]. \quad (2)$$

where $d_G(e) = d_G(u) + d_G(e) - 2$.

In [7], Kulli introduced the first and second K Hyper Banhatti indices and it was defined as

$$HB_1(G) = \sum_{ue} [d_G(u) + d_G(e)]^2. \quad (3)$$

$$HB_2(G) = \sum_{ue}[d_G(u)d_G(e)]^2. \quad (4)$$

Here, we introduce some new Topological indices based on eccentricity as follows.

If G is a graph with vertex set $V(G)$ and the edge set $E(G)$.Let $u \in V$(G) and $e \in E$(G) , define First and second K- Eccentric indices as,

$$B_1E(G) = \sum_{ue}\left[e_G(u) + e_{L(G)}(e)\right]. \quad (5)$$

$$B_2E(G) = \sum_{ue}\left[e_G(u)e_{L(G)}(e)\right]. \quad (6)$$

Similarly, First and second K Hyper -Eccentric indices are defined as,

$$HB_1E(G) = \sum_{ue} \left[e_G(u) + e_{L(G)}(e)\right]^2. \quad (7)$$

$$HB_2E(G) = \sum_{ue}\left[e_G(u)\, e_{L(G)}(e)\right]^2, \quad (8)$$

where in all the cases *ue* means that the vertex *u* and edge *e* are incident in *G* and $e_{L(G)}(e)$ is the eccentricity of e in the line graph $L(G)$ of G.

1. K-ECCENTRIC INDICES, K HYPER-ECCENTRIC INDICES, OF SOME SPECIAL GRAPHS.

Theorem 2.1 :

Let G be a complete graph K_n then

$$(i)\quad B_1E(K_n) = 3n(n-1). \quad (9)$$

$$B_2E(K_n) = 2n(n-1). \quad (10)$$

$$(ii)\quad HB_1E(K_n) = 9n(n-1). \quad (11)$$

$$HB_2E(K_n) = 4n(n-1). \quad (12)$$

Proof:

The complete graph K_n has n vertices, $m = \frac{n(n-1)}{2}$, edges and all the vertices has eccentricity 1. Also every vertex of K_n is incident with $(n-1)$ edges, that is "Every edge of K_n is incident with exactly 2 vertices ". Also $e_{L(K_n)}(e) = 2$.

(i) $B_1E(K_n) = \sum_{ue}\left[e_{K_n}(u) + e_{L(K_n)}(e)\right] = \sum_{\boldsymbol{u_i}} \sum_{\boldsymbol{e_j}}\left[e_{K_n}(\boldsymbol{u_i}) + e_{L(K_n)}(\boldsymbol{e_j})\right]$

$$\sum_{u_i}^{n}\sum_{e_j}^{n-1}[1+2] = \sum_{u_i}^{n} 3(n-1) = 3n(n-1).$$

$$Also\ \ B_2E(K_n) = \sum_{ue}\left[e_{K_n}(u)\, e_{L(K_n)}(e)\right] = \sum_{u_i}\sum_{e_j}\left[e_{K_n}(u_i)\, e_{L(K_n)}(e_j)\right]$$

$$= \sum_{u_i}^{n}\sum_{e_j}^{n-1}[1 \times 2] = \sum_{u_i}^{n} 2(n-1) = 2n(n-1).$$

(ii) $HB_1E(K_n) = \sum_{ue}\left[e_{K_n}(u) + e_{L(K_n)}(e)\right]^2$

$$= \sum_{\boldsymbol{u_i}} \sum_{\boldsymbol{e_j}} \left[e_{K_n}(\boldsymbol{u_i}) + e_{L(K_n)}(\boldsymbol{e_j})\right]^2$$

$$= \sum_{u_i}^{n} \sum_{e_j}^{n-1} [1+2]^2 = \sum_{u_i}^{n} 9(n-1) = 9\mathrm{n}(\mathrm{n}-1).$$

$$Also \quad HB_2E(K_n) = \sum_{ue} \left[e_{K_n}(u)\, e_{L(K_n)}(e)\right]^2$$

$$= \sum_{\boldsymbol{u_i}} \sum_{\boldsymbol{e_j}} \left[e_{K_n}(\boldsymbol{u_i})\, e_{L(K_n)}(\boldsymbol{e_j})\right]^2 = \sum_{u_i}^{n} \sum_{e_j}^{n-1} [1 \times 2]^2 = \sum_{u_i}^{n} 4(n-1) = 4\mathrm{n}(\mathrm{n}-1).$$

Theorem 2.2 :

Let G be a cycle graph C_n, then

$$(i) \quad B_1E(C_n) = \begin{cases} 2\mathrm{n}(n-1), n \text{ is odd} \\ 2\mathrm{n}^2, n \text{ is even.} \end{cases} \tag{13}$$

$$B_2E(C_n) = \begin{cases} \frac{\mathrm{n}(\mathrm{n}-1)^2}{2}, n \text{ is odd} \\ \frac{\mathrm{n}^3}{2}, \ n \text{ is even.} \end{cases} \tag{14}$$

$$(ii) \quad HB_1E(C_n) = \begin{cases} 2n(n-1)^2, n \text{ is odd} \\ 2\mathrm{n}^3, n \text{ is even.} \end{cases} \tag{15}$$

$$HB_2E(C_n) = \begin{cases} \frac{n}{8}(\mathrm{n}-1)^4, n \text{ is odd} \\ \frac{n^5}{8}, n \text{ is even.} \end{cases} \tag{16}$$

Proof:

Let C_n be a Cycle with $n \geq 3$ vertices. Every vertex of C_n is incident with exactly two edges, Every edge of C_n is incident with exactly two vertices. Eccentricity of any vertex in

$C_n = \begin{cases} \frac{n-1}{2}, \forall n \geq 3(\ n \text{ is odd}) \\ \frac{n}{2}, \forall n \geq 2(\ n \text{ is even}) \end{cases}$. Also $e_{L(C_n)}(e) = \begin{cases} \frac{n-1}{2}, n \text{ is odd} \\ \frac{n}{2}, n \text{ is even.} \end{cases}$

Proof of (i):

Case 1: If n is odd

$(i) \quad B_1E(C_n) = \sum_{ue}\left[e_{C_n}(u) + e_{L(C_n)}(e)\right] = \sum_{\boldsymbol{u_i}} \sum_{\boldsymbol{e_j}} \left[e_{C_n}(\boldsymbol{u_i}) + e_{L(C_n)}(\boldsymbol{e_j})\right]$

$$= \sum_{u_i}^{n} \sum_{e_j}^{2} \left[\frac{n-1}{2} + \frac{n-1}{2}\right] = \sum_{u_i}^{n} 2(n-1) = 2\mathrm{n}(n-1).$$

$Also \ \ B_2E(C_n) = \sum_{ue}\left[e_{C_n}(u)\, e_{L(C_n)}(e)\right] = \sum_{\boldsymbol{u_i}} \sum_{\boldsymbol{e_j}} \left[e_{C_n}(\boldsymbol{u_i}) \times e_{L(C_n)}(\boldsymbol{e_j})\right]$

$$=\sum_{u_i}^{n}\sum_{e_j}^{2}\left[\frac{n-1}{2}\times\frac{n-1}{2}\right]=\sum_{u_i}^{n}2\left[\frac{(n-1)^2}{4}\right]=\frac{\mathrm{n}(\mathrm{n}-1)^2}{2}.$$

Case 2: If n is even

$$(i)\quad B_1E(C_n)=\sum_{ue}\left[e_{C_n}(u)+e_{L(C_n)}(e)\right]=\sum_{\boldsymbol{u_i}}\sum_{\boldsymbol{e_j}}\left[e_{C_n}(\boldsymbol{u_i})+e_{L(C_n)}(\boldsymbol{e_j})\right]$$

$$=\sum_{u_i}^{n}\sum_{e_j}^{2}\left[\frac{n}{2}+\frac{n}{2}\right]=\sum_{u_i}^{n}2(n)=2\mathrm{n}^2.$$

$Also\ \ B_2E(C_n)=\sum_{ue}\left[e_{C_n}(u)\,e_{L(C_n)}(e)\right]=\sum_{\boldsymbol{u_i}}\sum_{\boldsymbol{e_j}}\left[e_{C_n}(\boldsymbol{u_i})\times e_{L(C_n)}(\boldsymbol{e_j})\right]$

$$=\sum_{u_i}^{n}\sum_{e_j}^{2}\left[\frac{n}{2}\times\frac{n}{2}\right]=\sum_{u_i}^{n}2\left[\frac{n^2}{4}\right]=\frac{\mathrm{n}^3}{2}.$$

Proof of (ii):

Case 1: If n is odd

$HB_1E(C_n)=\sum_{ue}\left[e_{C_n}(u)+e_{L(C_n)}(e)\right]^2=\sum_{\boldsymbol{u_i}}\sum_{\boldsymbol{e_j}}\left[e_{C_n}(\boldsymbol{u_i})+e_{L(C_n)}(\boldsymbol{e_j})\right]^2$

$$=\sum_{u_i}^{n}\sum_{e_j}^{2}\left[\frac{n-1}{2}+\frac{n-1}{2}\right]^2=\sum_{u_i}^{n}2[n-1]^2=2n(n-1)^2.$$

Also $$\mathrm{HB_2E(C_n)}=\sum_{\mathrm{ue}}\left[\mathrm{e_{C_n}(u)\,e_{L(C_n)}(e)}\right]^2=\sum_{\boldsymbol{u_i}}\sum_{\boldsymbol{e_j}}\left[e_{C_n}(\boldsymbol{u_i})e_{L(C_n)}(\boldsymbol{e_j})\right]^2$$

$$=\sum_{u_i}^{n}\sum_{e_j}^{2}\left[\frac{n-1}{2}\times\frac{n-1}{2}\right]^2=\sum_{u_i}^{n}2\left[\left(\frac{(n-1)^2}{4}\right)\right]^2=\frac{n}{8}(\mathrm{n}-1)^4.$$

Case 2: If n is even

$HB_1E(C_n)=\sum_{ue}\left[e_{C_n}(u)+e_{L(C_n)}(e)\right]^2=\sum_{\boldsymbol{u_i}}\sum_{\boldsymbol{e_j}}\left[e_{C_n}(\boldsymbol{u_i})+e_{L(C_n)}(\boldsymbol{e_j})\right]^2$

$$=\sum_{u_i}^{n}\sum_{e_j}^{2}\left[\frac{n}{2}+\frac{n}{2}\right]^2=\sum_{u_i}^{n}2[n]^2=2\mathrm{n}^3.$$

$Also\qquad HB_2E(C_n)=\sum_{ue}\left[e_{C_n}(u)\,e_{L(C_n)}(e)\right]^2$

$$=\sum_{\boldsymbol{u_i}}\sum_{\boldsymbol{e_j}}\left[e_{C_n}(\boldsymbol{u_i})\,e_{L(C_n)}(\boldsymbol{e_j})\right]^2==\sum_{u_i}^{n}\sum_{e_j}^{2}\left[\frac{n}{2}\times\frac{n}{2}\right]^2=\sum_{u_i}^{n}2\left[\left(\frac{n^2}{4}\right)\right]^2=\frac{n^5}{8}.$$

Theorem 2.3 :

Let G be a wheel graph W_n then

$$(i)\ \ B_1E(W_n) = 15(\mathrm{n}-1). \tag{17}$$
$$B_2E(W_n) = 14(\mathrm{n}-1). \tag{18}$$

$$(ii)\ \ HB_1E(W_n) = 57(\mathrm{n}-1). \tag{19}$$
$$HB_2E(W_n) = 52(\mathrm{n}-1). \tag{20}$$

Proof:

Let W_n be a wheel graph. The wheel graph W_n has n vertices and 2(n-1) edges. Every edge of W_n is incident with exactly two vertices. In W_n there are n-1 edges in centre with eccentricity 1and the remaining n-1 edges incident with centre u. Also we have $e(u) = 1, and\ e(u_i) = 2\ in\ G$. The vertex u_i is incident with exactly 3 edges. Also $e_{L(W_n)}(e) = 2$ for all edges.

Hence

$$(i)\ \ B_1E(W_n) = \sum_{ue}\left[e_{W_n}(u) + e_{L(W_n)}(e)\right]$$
$$= \sum_{\boldsymbol{u_i}}\sum_{\boldsymbol{e_j}}\left[e_{W_n}(\boldsymbol{u_i}) + e_{L(W_n)}(\boldsymbol{e_j})\right]$$
$$\sum_{\boldsymbol{j=1}}^{\boldsymbol{n-1}}\left[e_{W_n}(\boldsymbol{u}) + e_{L(W_n)}(\boldsymbol{e_j})\right] + \sum_{i=2}^{n}\sum_{\boldsymbol{e_j}}\left[e_{W_n}(\boldsymbol{u_i}) + e_{L(W_n)}(\boldsymbol{e_j})\right]$$
$$\sum_{\boldsymbol{u_i}}\sum_{\boldsymbol{e_j}}[(1+2)+(2+2)] = (\mathrm{n}-1)(1+2) + (\mathrm{n}-1)\times 3\times[2+2] = 15(\mathrm{n}-1).$$

$$Also\ B_2E(W_n) = \sum_{ue}\left[e_{W_n}(u)\ e_{L(W_n)}(e)\right]$$
$$\sum_{\boldsymbol{j=1}}^{\boldsymbol{n-1}}\left[e_{W_n}(\boldsymbol{u}) \times e_{L(W_n)}(\boldsymbol{e_j})\right] + \sum_{i=2}^{n}\sum_{\boldsymbol{e_j}}\left[e_{W_n}(\boldsymbol{u_i}) \times e_{L(W_n)}(\boldsymbol{e_j})\right]$$
$$\sum_{\boldsymbol{u_i}}\sum_{\boldsymbol{e_j}}[(1\times 2)+(2\times 2)] = (\mathrm{n}-1)(1\times 2) + (\mathrm{n}-1)\times 3\times[2\times 2] = 14(\mathrm{n}-1).$$

$$(ii)\ \ HB_1E(W_n) = \sum_{ue}\left[e_{W_n}(u) + e_{L(W_n)}(e)\right]^2$$
$$= \sum_{\boldsymbol{u_i}}\sum_{\boldsymbol{e_j}}\left[e_{W_n}(\boldsymbol{u_i}) + e_{L(W_n)}(\boldsymbol{e_j})\right]^2$$
$$\left[\sum_{\boldsymbol{j=1}}^{\boldsymbol{n-1}}\left[e_{W_n}(\boldsymbol{u}) + e_{L(W_n)}(\boldsymbol{e_j})\right] + \sum_{i=2}^{n}\sum_{\boldsymbol{e_j}}\left[e_{W_n}(\boldsymbol{u_i}) + e_{L(W_n)}(\boldsymbol{e_j})\right]\right]^{\boldsymbol{2}}$$

$$\sum_{u_i}\sum_{e_j}[(1+2)^2+(2+2)^2] = (\mathrm{n}-1)(1+2)^2+(\mathrm{n}-1)\times 3\times[2+2]^2 = 57(\mathrm{n}-1).$$

$$Also \qquad HB_2E(W_n) = \sum_{ue}\left[e_{W_n}(u)\, e_{L(W_n)}(e)\right]^2$$

$$= \sum_{\boldsymbol{u_i}}\sum_{\boldsymbol{e_j}}\left[e_{W_n}(\boldsymbol{u_i})\, e_{L(W_n)}(\boldsymbol{e_j})\right]^2$$

$$\left[\sum_{\boldsymbol{j=1}}^{\boldsymbol{n-1}}\left[e_{W_n}(\boldsymbol{u})\times e_{L(W_n)}(\boldsymbol{e_j})\right] + \sum_{i=2}^{n}\sum_{\boldsymbol{e_j}}\left[e_{W_n}(\boldsymbol{u_i})\times e_{L(W_n)}(\boldsymbol{e_j})\right]\right]^{\boldsymbol{2}}$$

$$(\mathrm{n}-1)(1\times 2)^2+(\mathrm{n}-1)\times 3\times[2\times 2]^2 = 52(\mathrm{n}-1)$$

$$= (\mathrm{n}-1)[4+16(\mathrm{n}-1)] = (\mathrm{n}-1)[16\mathrm{n}-12]$$

Theorem 2.4` :
Let G be a star graph S_n then

$$(i)\quad B_1E(S_n) = 5(n-1). \tag{21}$$
$$B_2E(S_n) = 3(n-1). \tag{22}$$

$$(ii)\quad HB_1E(S_n) = 13(n-1). \tag{23}$$
$$HB_2E(S_n) = 5(\mathrm{n}-1). \tag{24}$$

Proof:

The Star graph S_n has n vertices and n-1 edges. Every edge of S_n is incident with exactly two vertices. Let v be the central vertex ofS_n and $u_i, i = 1,2,3 \ldots n-1$ be other pendant vertices.The vertex v is incident with $n-1$ edges , u_i is incident with only one edge such that $e(v) = 1 and\ e(u) = 2$. Also $e_{L(S_n)}(e) = 1\ for\ e \in E(G)$.

Hence

$(i)\quad B_1E(S_n) = \sum_{ue}\left[e_{S_n}(u)+e_{L(S_n)}(e)\right] = \sum_{\boldsymbol{u_i}}\sum_{\boldsymbol{e_j}}\left[e_{S_n}(\boldsymbol{u_i})+e_{L(S_n)}(\boldsymbol{e_j})\right]$

$$\sum_{1}^{n-1}(1+1)+\sum_{\mathrm{u_i}}\sum_{\mathrm{e_j}}(2+1) = 2\,(\mathrm{n}-1)+\sum_{1}^{n-1}3 = 5(\mathrm{n}-1)$$

$$Also\ B_2E(S_n) = \sum_{ue}\left[e_{S_n}(u)\, e_{L(S_n)}(e)\right]$$

$$= \sum_{\mathrm{u_i}}\sum_{\mathrm{e_j}}\left[\mathrm{e}_{S_n}(\mathrm{u_i})\,\mathrm{e}_{L(S_n)}(\mathrm{e_j})\right] = \sum_{1}^{n-1}(1\times 1)+\sum_{\mathrm{u_i}}\sum_{\mathrm{e_j}}(2\times 1) = (\mathrm{n}-1)+\sum_{1}^{n-1}2 = 3(\mathrm{n}-1)$$

$$(ii)\quad HB_1E(S_n) = \sum_{ue}\left[e_{S_n}(u) + e_{L(S_n)}(e)\right]^2$$

$$= \sum_{\boldsymbol{u_i}}\sum_{\boldsymbol{e_j}}\left[e_{S_n}(\boldsymbol{u_i}) + e_{L(S_n)}(\boldsymbol{e_j})\right]^2$$

$$= \sum_{1}^{n-1}(1+1)^2 + \sum_{\mathrm{u_i}}\sum_{\mathrm{e_j}}(2+1)^2 = 4(\mathrm{n}-1) + \sum_{1}^{n-1} 9 = 13(\mathrm{n}-1)$$

$$Also \qquad HB_2E(S_n) = \sum_{ue}\left[e_{S_n}(u)\, e_{L(S_n)}(e)\right]^2$$

$$= \sum_{\boldsymbol{u_i}}\sum_{\boldsymbol{e_j}}\left[e_{S_n}(\boldsymbol{u_i})\, e_{L(S_n)}(\boldsymbol{e_j})\right]^2 = \sum_{1}^{n-1}(1\times 1)^2 + \sum_{\mathrm{u_i}}\sum_{\mathrm{e_j}}(2\times 1)^2 = (\mathrm{n}-1) + \sum_{1}^{n-1} 4 = 5(\mathrm{n}-1)$$

$$= 5(\mathrm{n}-1)$$

Theorem 2.5 :

Let G be a complete bipartite graph $K_{m,n}$, then

$$(i)\quad B_1E\left(K_{m,n}\right) = 8\mathrm{mn}. \qquad (25)$$

$$B_2E\left(K_{m,n}\right) = 8\mathrm{mn}. \qquad (26)$$

$$(ii)\quad HB_1E\left(K_{m,n}\right) = 32\mathrm{mn}. \qquad (27)$$

$$HB_2E\left(K_{m,n}\right) = 32\mathrm{mn}. \qquad (28)$$

Proof:

Let $K_{m,n}$ be a complete bipartite graph with $m+n$ vertices. Also $V = V_1 \cup V_2, |V_1| = m \; and \; |V_2| = n$.

Every vertex of V_1 is incident with *n* edges and every vertex of V_2 is incident with *m* edges. Let $V_1 = \{v_1, v_2, v_3 \dots v_m\}$ and $V_2 = \{w_1, w_2, w_3 \dots w_n\}$. The eccentricity of all vertices are 2. Also $e_{L(K_{m,n})}(e) = 2, for \; e \in (G)$.

Hence

$$(i)\quad B_1E\left(K_{m,n}\right) = \sum_{ue}\left[e_{K_{m,n}}(u) + e_{L(K_{m,n})}(e)\right]$$

$$= \sum_{\mathrm{v_i}}^{\mathrm{m}}\sum_{\mathrm{e_j}}^{\mathrm{n}}\left[e_{K_{m,n}}(\boldsymbol{v_i}) + e_{L(K_{m,n})}(\boldsymbol{e_j})\right] + \sum_{\mathrm{w_j}}^{\mathrm{n}}\sum_{\mathrm{e_i}}^{\mathrm{m}}\left[e_{K_{m,n}}(\mathrm{w_j}) + e_{L(K_{m,n})}(\mathrm{e_i})\right]$$

$$= \sum_{\mathrm{v_i}}^{\mathrm{m}}\sum_{\mathrm{e_j}}^{\mathrm{n}}[2+2] + \sum_{\mathrm{w_j}}^{\mathrm{n}}\sum_{\mathrm{e_i}}^{\mathrm{m}}[2+2] = \sum_{\mathrm{v_i}}^{\mathrm{m}}\mathrm{n}(4) + \sum_{\mathrm{w_j}}^{\mathrm{n}}\mathrm{m}(4)$$

$$= \mathrm{mn}(4) + \mathrm{mn}(4) = 8\mathrm{mn}$$

$$Also\; B_2E\left(K_{m,n}\right) = \sum_{ue}\left[e_{K_{m,n}}(u)\, e_{L(K_{m,n})}(e)\right]$$

$$= \sum_{v_i}^{m}\sum_{e_j}^{n}\left[e_{K_{m,n}}(\boldsymbol{v_i}) \times e_{L(K_{m,n})}(\boldsymbol{e_j})\right] + \sum_{w_j}^{n}\sum_{e_i}^{m}\left[e_{K_{m,n}}(\mathrm{w_j}) \times e_{L(K_{m,n})}(\mathrm{e_i})\right]$$

$$= \sum_{v_i}^{m}\sum_{e_j}^{n}[2 \times 2] + \sum_{w_j}^{n}\sum_{e_i}^{m}[2 \times 2] = \sum_{v_i}^{m} n(4) + \sum_{w_j}^{n} m(4)$$

$$= mn(4) + mn(4) = 8mn$$

$$(ii) \quad HB_1E(K_{m,n}) = \sum_{ue}\left[e_{K_{m,n}}(u) + e_{L(K_{m,n})}(e)\right]^2$$

$$= \sum_{v_i}^{m}\sum_{e_j}^{n}\left[e_{K_{m,n}}(\boldsymbol{v_i}) + e_{L(K_{m,n})}(\boldsymbol{e_j})\right]^2 + \sum_{w_j}^{n}\sum_{e_i}^{m}\left[e_{K_{m,n}}(\mathrm{w_j}) + e_{L(K_{m,n})}(\mathrm{e_i})\right]^2$$

$$= \sum_{v_i}^{m}\sum_{e_j}^{n}[2 + 2]^2 + \sum_{w_j}^{n}\sum_{e_i}^{m}[2 + 2]^2 = \sum_{v_i}^{m} n(16) + \sum_{w_j}^{n} m(16)$$

$$= mn(16) + mn(16) = 32mn$$

$$Also \qquad HB_2E(K_{m,n}) = \sum_{ue}\left[e_{K_{m,n}}(u)\, e_{L(K_{m,n})}(e)\right]^2$$

$$= \sum_{v_i}^{m}\sum_{e_j}^{n}\left[e_{K_{m,n}}(\boldsymbol{v_i}) \times e_{L(K_{m,n})}(\boldsymbol{e_j})\right]^2 + \sum_{w_j}^{n}\sum_{e_i}^{m}\left[e_{K_{m,n}}(\mathrm{w_j}) \times e_{L(K_{m,n})}(\mathrm{e_i})\right]^2$$

$$= \sum_{v_i}^{m}\sum_{e_j}^{n}[2 \times 2]^2 + \sum_{w_j}^{n}\sum_{e_i}^{m}[2 \times 2]^2 = \sum_{v_i}^{m} n(16) + \sum_{w_j}^{n} m(16)$$

$$= mn(16) + mn(16) = 32mn$$

Theorem 2.6 :

Let G be a Path graph P_n, then

$$(i) B_1E(P_n) = \begin{cases} 2[2r + (4r+3) + (4r+7) + (4r+11) + \cdots + (8r-9) + (4r-1)], n \text{ is odd} \\ 2[4r - 2 + (4r+3) + (4r+7) + (4r+11) + \cdots + (8r-9) + (4r-3)], n \text{ is even.} \end{cases}$$

(29)

$$B_2E(P_n) = \begin{cases} 2[2r^2 + (2r^2+3r+1) + (2r^2+7r+6) + \cdots + (8r^2-10r+3) + (8r^2-18r+10) + (4r^2-2r)], n \text{ is odd} \\ 2[(4r^2-4r+1) + (4r^2+6r+2) + (4r^2+14r+12) + \cdots + +(16r^2-36r+20) + (4r^2-6r+2)], n \text{ is even} \end{cases}$$

(30)

$$(ii) \quad HB_1E(P_n) = \begin{cases} 2[2r^2 + (4r+3)^2 + (4r+7)^2 + \cdots + (8r-5)^2 + (8r-9)^2 + (4r-1)^2], n \text{ is odd .} \\ 2[(4r-2)^2 + (4r+3)^2 + (4r+7)^2 + (4r+11)^2 + \cdots + (8r-9)^2 + (4r-3)^2], n \text{ is even} \end{cases}$$

(31)

$$HB_2E(P_n) = \begin{cases} 2[8r^4 + (2r^2+3r+1)^2 + (2r^2+7r+6)^2 + \cdots + (4r^2-10r+6)^2 + (4r^2-2r)^2], n \text{ is odd} \\ 2[(4r^2-4r+1)^2 + (4r^2+6r+2)^2 + (4r^2+14r+12)^2 + \cdots + (16r^2-36r+20)^2 + (16r^2-8r+9)^2], n \text{ is ev} \end{cases}$$

(32)

Proof:

Proof of (i):

Case 1: If n is odd

Let r be the radius of P_n. Then $r = \frac{n-1}{2}$ and $n = 2r+1$. Then

P_n has only one unicentral vertex with eccentricity r,

P_n has two vertices with eccentricity r+1,

……………………

P_n has two vertices with eccentricity 2r,

In this case $L\,(P_n) = P_{n-1}, n-1 = 2r$.

P_{n-1} has 2 central vertices with eccentricity r, 2 vertices with eccentricity $r+1$,…… 2 vertices with eccentricity $2r-1$.

G has only one vertex with $e(v) = r$ and it has two incident edges with $e_{L(G)}(e) = r$;

2 vertices with eccentricity $e(v) = r+1$ and each vertex has 2 incident edges with $e_{L(G)}(e) = r \text{ and } r+1$,……… and 2 peripheral vertices with eccentricity $2r$ with one incident edge having eccentricity $2r-1$.

$$(i) B_1E(P_n) = \sum_{ue}\left[e_{P_n}(u) + e_{L(P_n)}(e)\right] = \sum_{e_j}\sum_{\boldsymbol{u_i}}\left[e_{P_n}(\boldsymbol{u_i}) + e_{L(p_n)}(\boldsymbol{e_j})\right]$$
$$= 1[(r+r)+(r+r)] + 2\big[\big((r+1)+r\big) + \big((r+1)+(r+1)\big)\big]$$
$$+ 2\big[\big((r+2)+(r+1)\big) + \big((r+2)+(r+2)\big)\big]$$
$$+ \cdots + 2\big[\big((2r-1)+(2r-2)\big) + \big((2r-1)+(2r-1)\big)\big] + 2\big[\big((2r-3)+(2r-2)\big) + \big((2r-2)+(2r-2)\big)\big] + 2[2r+2r-1].$$
$$= 2[2r + (4r+3) + (4r+7) + (4r+11) + \cdots + (8r-9) + (4r-1)].$$

$$B_2E(P_n) = \sum_{ue}\left[e_{P_n}(u) \times e_{L(P_n)}(e)\right] = \sum_{e_j}\sum_{u_i}\left[e_{P_n}(u_i) \times e_{L(p_n)}(e_j)\right]$$
$$= 1[(r+r)\times(r+r)] + 2\big[\big((r+1)\times r\big) + \big((r+1)\times(r+1)\big)\big]$$
$$+ 2\big[\big((r+2)\times(r+1)\big) + \big((r+2)\times(r+2)\big)\big] + \cdots$$
$$+ 2\big[\big((2r-1)\times(2r-2)\big) + \big((2r-1)\times(2r-1)\big)\big]$$
$$+ 2\big[\big((2r-3)\times(2r-2)\big) + \big((2r-2)\times(2r-2)\big)\big] + 2[2r \times 2r-1]$$
$$= 2[2r^2 + (2r^2+3r+1) + (2r^2+7r+6) + \cdots + (8r^2-10r+3) + (8r^2-18r+10) + (4r^2-2r)]$$

Case 2: If n is even

Let r be the radius of P_n. Then $r = \frac{n}{2}$ and $n = 2r$. Then

P_n has two central vertices with eccentricity r,

P_n has two vertices with eccentricity r+1,

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……………………………
P_n has two vertices with eccentricity 2r-1,

In this case $L\ (P_n) = P_{n-1}, n-1 = 2r-1$.

P_{n-1} has only one vertex with eccentricity $r-1$, 2 vertices with eccentricity r,………… 2 vertices with eccentricity $2r-2$.

G has two vertices with $e(v) = r$ and each one has two incident edges such that one edge has eccentricity $r-1\ in\ L(G)$ and another one has eccentricity $r\ in\ L(G)$

2 vertices with eccentricity $e(v) = r+1$ and each vertex has 2 incident edges with $e_{L(G)}(e) = r\ and\ r+1$,…………………… and 2 peripheral vertices with eccentricity $2r-1$ with incident edges with $e_{L(G)}(e) = 2r-2$.

$$(i) B_1E(P_n) = \sum_{ue}\left[e_{P_n}(u) + e_{L(P_n)}(e)\right] = \sum_{e_j}\sum_{\boldsymbol{u_i}}\left[e_{P_n}(\boldsymbol{u_i}) + e_{L(p_n)}(\boldsymbol{e_j})\right]$$
$$= 2[(r+r-1)+(r+r-1)] + 2\left[\left((r+1)+r\right)+\left((r+1)+(r+1)\right)\right] + 2\left[\left((r+2)+(r+1)\right)+\left((r+2)+(r+2)\right)\right]$$
$$+\cdots+2\left[\left((2r-1)+(2r-2)\right)+\left((2r-1)+(2r-1)\right)\right]+2\left[\left((2r-3)+(2r-2)\right)+\left((2r-2)+(2r-2)\right)\right]+2[2r-1+2r-2].$$
$$= 2[4r-2+(4r+3)+(4r+7)+(4r+11)+\cdots+(8r-9)+(4r-3)].$$

$$B_2E(P_n) = \sum_{ue}\left[e_{P_n}(u) \times e_{L(P_n)}(e)\right] = \sum_{e_j}\sum_{u_i}\left[e_{P_n}(u_i) \times e_{L(p_n)}(e_j)\right]$$
$$= 2[(r+r-1)\times(r+r-1)] + 2\left[\left((r+1)+r\right)\times\left((r+1)+(r+1)\right)\right] + 2\left[\left((r+2)+(r+1)\right)\times\left((r+2)+(r+2)\right)\right]$$
$$+\cdots+2\left[\left((2r-1)+(2r-2)\right)\times\left((2r-1)+(2r-1)\right)\right]+2\left[\left((2r-3)+(2r-2)\right)\times\left((2r-2)+(2r-2)\right)\right]+2[2r-1\times 2r-2].$$
.
$$= 2[(4r^2-4r+1)+(4r^2+6r+2)+(4r^2+14r+12)+\cdots+(16r^2-14r+6) + (16r^2-36r+20)+(4r^2-6r+2)]$$

Proof of (ii):

Case 1: If n is odd

Let r be the radius of P_n. Then $r = \frac{n-1}{2}$ and $n = 2r+1$. Then

P_n has only one unicentral vertex with eccentricity r,

P_n has two vertices with eccentricity r+1,

……………………………
P_n has two vertices with eccentricity 2r,

In this case $L\ (P_n) = P_{n-1}, n-1 = 2r$.

P_{n-1} has 2 central vertices with eccentricity r, 2 vertices with eccentricity $r+1$,…… 2 vertices with eccentricity $2r-1$.

G has only one vertex with $e(v) = r$ and it has two incident edges with $e_{L(G)}(e) = r$;

2 vertices with eccentricity $e(v) = r+1$ and each vertex has 2 incident edges with

$e_{L(G)}(e) = r \; and \; r+1$,……… and 2 peripheral vertices with eccentricity $2r$ with one incident edge having eccentricity $2r-1$.

$HB_1E(P_n) = \sum_{ue}\left[e_{P_n}(u) + e_{L(P_n)}(e)\right]^2 = \sum_{\boldsymbol{u_i}}\sum_{\boldsymbol{e_j}}\left[e_{P_n}(\boldsymbol{u_i}) + e_{L(P_n)}(\boldsymbol{e_j})\right]^2$

$= 1[(r+r)+(r+r)]^2 + 2\left[\left((r+1)+r\right)+\left((r+1)+(r+1)\right)\right]^2 + 2\left[\left((r+2)+(r+1)\right)+\left((r+2)+(r+2)\right)\right]^2 + \cdots + 2\left[\left((2r-1)+(2r-2)\right)+\left((2r-1)+(2r-1)\right)\right]^2 + 2\left[\left((2r-3)+(2r-2)\right)+\left((2r-2)+(2r-2)\right)\right] + 2[2r+2r-1]^2.$

$$= 2[2r^2 + (4r+3)^2 + (4r+7)^2 + \cdots + (8r-5)^2 + (8r-9)^2 + (4r-1)^2]$$

$$\text{Also} \qquad HB_2E(P_n) = \sum_{ue}\left[e_{P_n}(u)\, e_{L(P_n)}(e)\right]^2 = \sum_{\boldsymbol{u_i}}\sum_{\boldsymbol{e_j}}\left[e_{P_n}(\boldsymbol{u_i}) e_{L(P_n)}(\boldsymbol{e_j})\right]^2$$

$1[(r+r)\times(r+r)]^2 + 2\left[\left((r+1)\times r\right)+\left((r+1)\times(r+1)\right)\right]^2 + 2\left[\left((r+2)\times(r+1)\right)+\left((r+2)\times(r+2)\right)\right]^2 + \cdots + 2\left[\left((2r-1)\times(2r-2)\right)+\left((2r-1)\times(2r-1)\right)\right]^2 + 2\left[\left((2r-3)\times(2r-2)\right)+\left((2r-2)\times(2r-2)\right)\right]^2 + 2[2r\times(2r-1)]^2.$

$$= 2[8r^4 + (2r^2+3r+1)^2 + (2r^2+7r+6)^2 + \cdots + (4r^2-10r+6)^2 + (4r^2-2r)^2]$$

<u>Case 2:</u> If n is even

Let r be the radius of P_n. Then $r = \frac{n}{2}$ and $n = 2r$. Then

P_n has two central vertices with eccentricity r,

P_n has two vertices with eccentricity r+1,

………………………………

P_n has two vertices with eccentricity 2r-1,

In this case $L\,(P_n) = P_{n-1}, n-1 = 2r-1$.

P_{n-1} has only one vertex with eccentricity $r-1$, 2 vertices with eccentricity r,………… 2 vertices with eccentricity $2r-2$.

G has two vertices with $e(v) = r$ and each one has two incident edges such that one edge has eccentricity $r-1 \; in \; L(G)$ and another one has eccentricity $r \; in \; L(G)$

2 vertices with eccentricity $e(v) = r+1$ and each vertex has 2 incident edges with $e_{L(G)}(e) = r \; and \; r+1$,…………………… and 2 peripheral vertices with eccentricity $2r-1$ with incident edges with $e_{L(G)}(e) = 2r-2$.

$HB_1E(P_n) = \sum_{ue}\left[e_{P_n}(u) + e_{L(P_n)}(e)\right]^2 = \sum_{\boldsymbol{u_i}}\sum_{\boldsymbol{e_j}}\left[e_{P_n}(\boldsymbol{u_i}) + e_{L(P_n)}(\boldsymbol{e_j})\right]^2$

$$= 2[(r+r-1)+(r+r-1)]^2 + 2\left[\left((r+1)+r\right)+\left((r+1)+(r+1)\right)\right]^2 + 2\left[\left((r+2)+(r+1)\right)+\left((r+2)+(r+2)\right)\right]$$

$+ \cdots + 2\left[\left((2r-1)+(2r-2)\right)+\left((2r-1)+(2r-1)\right)\right]^2 + 2\left[\left((2r-3)+(2r-2)\right)+\left((2r-2)+(2r-2)\right)\right]^2 + 2[2r-1+2r-2]^2.$

$$= 2[(4r-2)^2 + (4r+3)^2 + (4r+7)^2 + (4r+11)^2 + \cdots + (8r-9)^2 + (4r-3)^2]$$

Also $\quad \mathrm{HB_2E}(P_n) = \sum_{\mathrm{ue}} \left[\mathrm{e}_{P_n}(\mathrm{u})\, \mathrm{e}_{\mathrm{L}(P_n)}(\mathrm{e})\right]^2 = \sum_{\boldsymbol{u_i}} \sum_{\boldsymbol{e_j}} \left[e_{P_n}(\boldsymbol{u_i}) e_{L(P_n)}(\boldsymbol{e_j})\right]^2$

$= \quad 2[(\mathrm{r}+\mathrm{r}-1) \times (\mathrm{r}+\mathrm{r}-1)]^2 + 2\left[\left((\mathrm{r}+1)+\mathrm{r}\right) \times \left((\mathrm{r}+1)+(\mathrm{r}+1)\right)\right]^2 + 2\left[\left((\mathrm{r}+2)+(\mathrm{r}+1)\right) \times \left((\mathrm{r}+2)+(\mathrm{r}+2)\right)\right] + \cdots + 2\left[\left((2\mathrm{r}-1)+(2\mathrm{r}-2)\right) \times \left((2\mathrm{r}-1)+(2\mathrm{r}-1)\right)\right]^2 + 2\left[\left((2\mathrm{r}-3)+(2\mathrm{r}-2)\right) \times \left((2\mathrm{r}-2)+(2\mathrm{r}-2)\right)\right]^2 + 2[2\mathrm{r}-1+2\mathrm{r}-2]^2.$

$$= 2[(4r^2-4r+1)^2 + (4r^2+6r+2)^2 + (4r^2+14r+12)^2 + \cdots + (16r^2-20r+6)^2 + (16r^2-36r+20)^2 + (16r^2-8r+9)^2]$$

Reference:

[1] K. C. Das, D. W. Lee and A. Graovac, Some properties of the Zagreb eccentricity indices, Ars Math. Contemp., 6 (2013) 117-125.

[2] N. De, On multiplicative Zagreb eccenticity indices, South Asian J. Math., 2 no. 6 (2012) 570-577.

[3] Z. Du, B. Zhou and N. Trinajsti_c, Extremal properties of the Zagreb eccentricity indices, Croat. Chem. Acta, 85 no. 3 (2012), 359-362.

[4] D. Vukiˇcevi´c and A. Graovac, Note on the comparison of the first and second normalized Zagreb eccentricity indices, Acta Chim. Slov. 57 (2010), 524–528.

[5] R. Todeschini and V. Consonni, New local vertex invariants and molecular descriptors based on functions of the vertex degree, MATCH Commun. Math. Comput. Chem., 64 (2010) 359-372.

[6] V. R. Kulli, On *K* Banhatti Indices of Graphs, Journal of Computer and Mathematical Sciences, *Vol.7(4), 213-218, April 2016*

[7] V. R. Kulli , On K hyper-Banhatti indices and Coindices of graphs Research gate *1-6, 11 november 2016.*

[8] V. R. Kulli, First Multiplicative *K* Banhatti Index and Coindex of Graphs, *Annals of Pure and Applied Mathematics Vol. 11, No. 2, 2016, 79-82.*

[9] V. R. Kulli, Second Multiplicativee *K* Banhatti Index and Coindex of Graphs, Journal of Computer and Mathematical Sciences, *Vol.7(5), 254-258, May 2016.*

[10] V. R. Kulli , Multiplicative K hyper-Banhatti indices and coindices of graphs, International Journal of Mathematical Archive, *1-6*, June, (2016).