Optimal Homotopy Asymptotic Method With Different Auxiliary Functions For The Solution Of Seventh Order Boundary Value Problems

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Abstract:

In this work, we solve seventh-order boundary value problems by optimal homotopy asymptotic method using different types of auxiliary functions. Approximations are obtained for different auxiliary functions in terms of convergent series with excellent accuracy. Three numerical examples have been given to show comparisons of numerical results for different auxiliary functions.

Keywords: Approximate solution, Auxiliary function, Galerkin's method, Linear and nonlinear boundary value problems, Optimal homotopy asymptotic method.

1. Introduction

The behavior of Induction motor is commonly simulated by fifth order boundary value problems. Seventh order problems arise in the modeling of induction motors where the rotor frequency is not a single value. The model includes two stator state variables, two rotor state variables, and shaft speed. Two more state variables must be added to put together the effects of a second rotor circuit representing deep bars, a starting cage, or rotor distributed parameters. Under transient conditions, when the rotor frequency is not a single value, the model is simulated by seventh order boundary problems [1].Many authors presented the numerical solutions of seventh order boundary value problems. Siddiqi *et al*. [2-5] used variational iteration method (VIM), adomian decomposition method (ADM), differential transform method (DTM) and optimal homotopy asymptotic method (OHAM) to solve seventh order boundary value problems.The optimal homotopy asymptotic method is the newest of all the aforementioned methods. It has developed by Vasile Marinca et al. [6-9]. This method provides a series solution. Convergence of this solution is controlled by the auxiliary constants which appear in the auxiliary function. Different auxiliary functions can be chosen to produce excellent results. The algorithm of OHAM is composed of explicit relations and only a few iterations produce excellent approximations. It also allows adjustment of convergence region where it is needed. One of the biggest advantage of OHAM is that there is no need to confirm the results of your solution by applying some other method because the residual serves as an indicator for the error.

Javed Ali et al. [10-12], used OHAM for multi-point and two point boundary value problems.

2. Analysis of the Optimal Homotopy Asymptotic Method

Consider the following boundary value problem:

$$\begin{aligned}&\Lambda(\psi(\xi))+\gamma(\xi)+\mathrm{N}(\psi(\xi))=0,\\&\mathrm{B}\left(\psi,\frac{\delta\psi}{\delta\xi}\right)=0,\end{aligned}\qquad(1)$$

where Λ is a linear operator, ξ denotes independent variable, $\psi(\xi)$ is an unknown function, $\gamma(\xi)$ is a known function, N is a nonlinear operator and B is a boundary operator.

According to OHAM we construct a homotopy $\mathrm{H}(\upsilon, p):\Box\times[0,1]\rightarrow\Box$ which satisfies

$$\begin{aligned}&(1-p)[\Lambda(\upsilon)+\gamma(\xi)]=h(p)[\Lambda(\upsilon)+\gamma(\xi)+\mathrm{N}(\upsilon)],\\&\mathrm{B}\left(\upsilon,\frac{\partial\upsilon}{\partial\xi}\right)=0,\end{aligned}\qquad(2)$$

where $p\in[0,1]$ is an embedding parameter, $h(p)\neq 0$ for $p\neq 0$ and $h(0)=0$. Obviously, when $p=0$ and $p=1$, it holds that $\upsilon(\xi,0)=\psi_0(\xi)$ and $\upsilon(\xi,1)=\psi(\xi)$ respectively.

Thus, as p approaches from 0 to 1, the solution $\upsilon(\xi, p)$ approaches from $\psi_0(\xi)$ to $\psi(\xi)$, where $\psi_0(\xi)$ is obtained from Eqn. (3) that is obtained from Eqn. (2) for $p=0$.

$$\begin{aligned}&\Lambda\left(\psi_0(\xi)\right)+\gamma(\xi)=0,\Box\\&\mathrm{B}\left(\psi_0,\frac{\delta\psi_0}{\delta\xi}\right)=0.\end{aligned}\qquad(3)$$

Next we choose the auxiliary function $h(p)$ in the form: $h(p)=pC_1\phi_1(\xi)+p^2C_2\phi_2(\xi)+...$ or $h(p)=p(C_1\phi_1(\xi)+C_2\phi_2(\xi)+...)$.

Any auxiliary function can be used to develop a general algorithm for OHAM. We take the former one for this purpose. i.e., $h(p)$ as $h(p)=pC_1+p^2C_2+\ ...,$ keeping $\phi_i(\xi)=1\ \forall\ i$.

In these auxiliary functions C_1 , $C_2 + \ldots$ are known as convergence control parameters and they are to be determined. To get an approximate solution, we expand $\upsilon(\xi, p, C_i)$ in Taylor's series about p in the following manner:

$$\upsilon(\xi, p, C_i) = \psi_0(\xi) + \sum_{\kappa=1}^{\infty} \psi_\kappa(\xi, C_1, C_2, \ldots, C_\kappa)\, p^\kappa. \tag{4}$$

Substituting Eqn. (4) and the auxiliary function $h(p) = pC_1 + p^2 C_2 + \ldots$ into Eqn. (2) and then equating the coefficient of like powers of p, we obtain the following linear equations:

Zeroth order problem is given by Eqn. (3) and the first order problem is given by Eqn. (5):

$$\Lambda(\psi_1(\xi)) + \gamma(\xi) = C_1 \mathrm{N}_0(\psi_0(\xi)),$$
$$\mathrm{B}\left(\psi_1, \frac{\delta \psi_1}{\delta \xi}\right) = 0. \tag{5}$$

The general governing equations for $\upsilon_\kappa(\xi)$ are given by:

$$\Lambda(\psi_\kappa(\xi)) - \Lambda(\psi_{\kappa-1}(\xi)) =$$
$$C_\kappa \mathrm{N}_0(\psi_0(\xi)) + \sum_{i=1}^{\kappa-\lambda} C_i \left[\Lambda(\psi_{\kappa-i}(\xi)) + \mathrm{N}_{\kappa-i}(\psi_0(\xi), \psi_1(\xi), \ldots, \psi_{\kappa-\lambda}(\xi))\right], \ \kappa = 2, 3, \ldots, \tag{6}$$
$$\mathrm{B}\left(\psi_\kappa, \frac{\delta \psi_\kappa}{\delta \xi}\right) = 0,$$

where $\mathrm{N}_m(\psi_0(\xi), \psi_1(\xi), \ldots, \psi_m(\xi))$ is the coefficient of p^m in the expansion of $\mathrm{N}(\upsilon)$ about the embedding parameter p.

$$\mathrm{N}(\upsilon(\xi, p, C_i)) = \mathrm{N}_0(\psi_0(\xi)) + \sum_{m=1}^{\infty} \mathrm{N}_m(\psi_0, \psi_1, \psi_2, \ldots, \psi_m)\, p^m. \tag{7}$$

It has been observed that the convergence of the series (4) depends upon the auxiliary parameters C_1 , $C_2, \ldots$.

If it is convergent at $p = 1$, one has

$$\upsilon(\xi, C_i) = \psi(\xi, C_i) = \psi_0(\xi) + \sum_{\kappa=1}^{\infty} \psi_\kappa(\xi, C_1, C_2, \dots, C_\kappa). \tag{8}$$

The result of the mth-order approximations are given by

$$\tilde{\psi}(\xi, C_1, C_2, \dots, C_m) = \psi_0(\xi) + \sum_{\kappa=1}^{m} \psi_\kappa(\xi, C_1, C_2, \dots, C_\kappa). \tag{9}$$

Substituting Eqn. (9) into Eqn. (1), it results the following residual:

$$R(\xi, C_1, C_2, \dots, C_m) = \Lambda(\tilde{\psi}(\xi, C_1, C_2, \dots, C_m)) + \gamma(\xi) + \mathrm{N}(\tilde{\psi}(\xi, C_1, C_2, \dots, C_m)). \tag{10}$$

If $R = 0$, then $\tilde{\psi}$ will be the exact solution. Generally it doesn't happen, especially in nonlinear problems.

In order to find the optimal values of C_i, $i = 1, 2, 3, \dots$, we first construct the functional,

$$\vartheta(C_1, C_2, \dots, C_m) = \int_a^b R^2(\xi, C_1, C_2, \dots, C_m)\, d\xi. \tag{11}$$

and then minimizing it, we have

$$\frac{\partial \vartheta}{\partial C_1} = \frac{\partial \vartheta}{\partial C_2} = \dots = \frac{\partial \vartheta}{\partial C_m} = 0, \tag{12}$$

where a and b are in the domain of the problem. With these of the parameters known, the approximate solution (of order m) is well-determined. This method is known as the method of least squares.

Alternatively we can solve the following system that is obtained by applying Galerkin's method to find C_i, $i = 1, 2, 3, \dots$.

$$\int_a^b R \frac{\partial \tilde{\psi}}{\partial C_i} d\xi = 0. \tag{13}$$

This procedure requires lesser computational cost than the method of least squares.

3. Numerical Examples

Example 3.1

Consider the following seventh order linear boundary value problem:

$$\psi^{(7)}(\xi)=\psi(\xi)-7Exp(\xi),\ 0<\xi<1,$$
$$\psi(0)=1,\ \psi^{(1)}(0)=0,\ \psi^{(2)}(0)=-1,\ \psi^{(3)}(0)=-2,$$
$$\psi(1)=0,\ \psi^{(1)}(1)=-Exp(1),\ \psi^{(2)}(1)=-2Exp(1).$$

Exact solution of this problem is $\psi(\xi)=(1-\xi)Exp(\xi).$

We want to get approximations of OHAM for the auxiliary function $h(p)=pC_1+p^2C_2$.

Following the rest of procedure of OHAM, the following problems are obtained:

Zeroth order problem:

$$\begin{aligned}&\psi_0^{(7)}(\xi)=0,\\&\psi_0(0)=1,\ \psi_0'(0)=0,\ \psi_0''(0)=-1,\ \psi_0'''(0)=-2\\&\psi_0(1)=0,\ \psi_0'(1)=-Exp(1),\ \psi_0''(1)=-2Exp(1)\end{aligned} \tag{14}$$

First order problem:

$$\begin{aligned}&\psi_1^{(7)}(\xi,C_1)=C_1(7\,Exp(\xi)-\psi_0(\xi))+(1+C_1)\psi_0^{(7)}(\xi),\\&\psi_1(0)=0,\ \psi_1'(0)=0,\ \psi_1''(0)=0,\ \psi_1'''(0)=0\\&\psi_1(1)=0,\ \psi_1'(1)=0,\ \psi_1''(1)=0.\end{aligned} \tag{15}$$

Second order problem:

$$\begin{aligned}&\psi_2^{(7)}(\xi,C_1,C_2)=C_2(7\,Exp(\xi)-\psi_0(\xi)+\psi_0^{(7)}(\xi))-C_1\psi_1(\xi,C_1)+(1+C_1)\psi_1^{(7)}(\xi,C_1),\\&\psi_2(0)=0,\ \psi_2'(0)=0,\ \psi_2''(0)=0,\ \psi_0'''(0)=0\\&\psi_2(1)=0,\ \psi_2'(1)=0,\ \psi_2''(1)=0.\end{aligned} \tag{16}$$

Using Eqns. (14), (15), and (16), the second order approximate solution for $p=1$ is:

$$\tilde{\psi}(\xi)=\psi_0(\xi)+\psi_1(\xi,C_1)+\psi_2(\xi,C_1,C_2)+O(\xi^{14}).$$

Following the procedure (13) described in Section 2 for determination of optimal values of $C's$ on the domain between $a=0$ and $b=1$, we find the following values:

$C_1 = 0$ & $C_2 = -1.000005848.$

By considering these values, the approximate solution becomes:

$$\tilde{\psi}(\xi) = 1 - \xi^2/2 - \xi^3/3 - 0.1250000\,\xi^4 - 0.033333\,\xi^5 - 0.0069444\,\xi^6 - 0.001190483\,\xi^7 - 0.000173612\xi^8 - 0.000022046\xi^9 - 2.4802\times10^{-6}\xi^{10} - 2.5165\times10^{-7}\xi^{11} - 2.1622\times10^{-8}\xi^{12} - 2.491\times10^{-9}\xi^{13} + O(\xi^{14}). \quad (17)$$

Let us consider now the auxiliary function $h(p) = p(C_1\phi_1(\xi) + C_2\phi_2(\xi))$. We take $\phi_1(\xi) = 1$ and $\phi_2(\xi) = \xi$, and obtain the following directly integrable equations:

Zeroth order problem:

$$\psi_0^{(7)}(\xi) = 0,$$
$$\psi_0(0) = 1,\ \psi_0'(0) = 0,\ \psi_0''(0) = -1,\ \psi_0'''(0) = -2$$
$$\psi_0(1) = 0,\ \psi_0'(1) = -Exp(1),\ \psi_0''(1) = -2Exp(1)$$

First order problem:

$$\psi_1^{(7)}(\xi, C_1) = 7Exp(\xi)(C_1 + C_2\xi) - (C_1 + C_2\xi)\psi_0(\xi) + (1 + C_1 + C_2\xi)\psi_0^{(7)}(\xi),$$
$$\psi_1(0) = 0,\ \psi_1'(0) = 0,\ \psi_1''(0) = 0,\ \psi_1'''(0) = 0$$
$$\psi_1(1) = 0,\ \psi_1'(1) = 0,\ \psi_1''(1) = 0.$$

Second order problem:

$$\psi_2^{(7)}(\xi, C_1, C_2) = (1 + C_1 + C_2\xi)\psi_1^{(7)}(\xi, C_1) - (C_1 + C_2\xi)\psi_1(\xi, C_1),$$
$$\psi_2(0) = 0,\ \psi_2'(0) = 0,\ \psi_2''(0) = 0,\ \psi_0'''(0) = 0$$
$$\psi_2(1) = 0,\ \psi_2'(1) = 0,\ \psi_2''(1) = 0.$$

The second order approximate solution is given by:

$$\tilde{\psi}(\xi) = \psi_0(\xi) + \psi_1(\xi, C_1) + \psi_2(\xi, C_1, C_2) + O(\xi^{14}).$$

Using the same procedure for C's we get, $C_1 = -1.00037658$ & $C_2 = 0.00073941.$

By considering these values, the approximate solution becomes:

$$\begin{aligned}\tilde{\psi}(\xi) &= 1-\xi^2/2-\xi^3/3-0.12500000\,\xi^4-0.03333333\,\xi^5-0.00694445\,\xi^6\\&-0.00119048\,\xi^7-0.00017361\xi^8-0.00002205\xi^9-2.4802\times10^{-6}\,\xi^{10}\\&-2.5052\times10^{-7}\,\xi^{11}-2.2965\times10^{-8}\,\xi^{12}-1.9263\times10^{-9}\,\xi^{13}+O(\xi^{14}).\end{aligned} \tag{18}$$

Table 1 shows the error obtained by using the two types of auxiliary functions for different values of ξ in the domain of the problem.

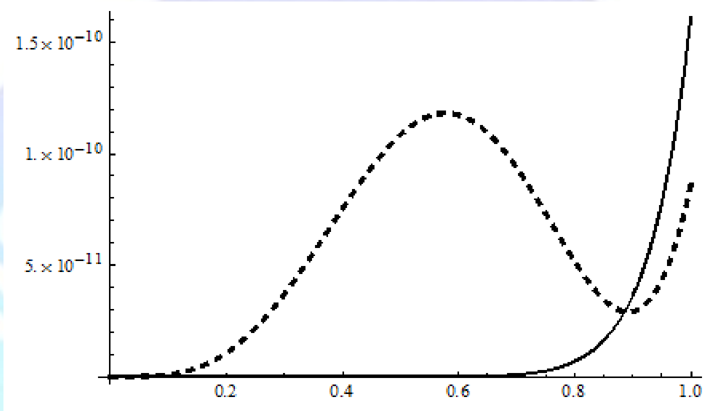
Table 1: Comparison of numerical results for problem 3.1

ξ	Absolute Error OHAM (17)	Absolute Error OHAM (18)
0.0	0	0
0.1	9.4180E-13	2.7756E-15
0.2	1.0784E-11	2.7645E-14
0.3	3.7256E-11	8.8041E-14
0.4	7.5508E-11	1.6576E-13
0.5	1.0859E-10	2.1661E-13
0.6	1.1732E-10	1.1313E-13
0.7	9.3709E-11	8.8174E-13
0.8	5.1531E-11	6.8708E-12
0.9	2.9246E-11	3.6492E-11
1.0	8.6004E-11	1.6082E-10

Error = Exact – Approximate

Figure 1: Dashed – Absolute error curve for approximate solution (17)

Solid- Absolute error curve for approximate solution (18)



Example 3.2

Consider the following seventh order linear boundary value problem:

$$\psi^{(7)}(\xi)=\xi\psi(\xi)+(\xi^2-2\xi-6)Exp(\xi),\ 0<\xi<1,$$
$$\psi(0)=1,\ \psi^{(1)}(0)=0,\ \psi^{(2)}(0)=-1,\ \psi^{(3)}(0)=-2,$$
$$\psi(1)=0,\ \psi^{(1)}(1)=-Exp(1),\ \psi^{(2)}(1)=-2Exp(1).$$

Exact solution of this problem is $\psi(\xi)=(1-\xi)Exp(\xi)$.

We use the usual auxiliary function $h(p)=pC_1+p^2C_2$.

Following the rest of procedure of OHAM, the following linear problems are obtained:

Zeroth order problem:

$$\begin{aligned}&\psi_0{}^{(7)}(\xi)=0,\\ &\psi_0(0)=1,\ \psi_0'(0)=0,\ \psi_0''(0)=-1,\ \psi_0'''(0)=-2 \qquad (19)\\ &\psi_0(1)=0,\ \psi_0'(1)=-Exp(1),\ \psi_0''(1)=-2Exp(1)\end{aligned}$$

First order problem:

$$\psi_1^{(7)}(\xi, C_1) = C_1 Exp(\xi)(6+2\xi-\xi^2) - C_1\xi\psi_0(\xi) + (1+C_1)\psi_0^{(7)}(\xi),$$
$$\psi_1(0)=0,\ \psi_1'(0)=0,\ \psi_1''(0)=0,\ \psi_1'''(0)=0 \quad (20)$$
$$\psi_1(1)=0,\ \psi_1'(1)=0,\ \psi_1''(1)=0.$$

Second order problem:

$$\psi_2^{(7)}(\xi, C_1, C_2) = C_2 Exp(\xi)(6+2\xi-\xi^2) - C_2\xi\psi_0(\xi)$$
$$-C_1\xi\psi_1(\xi, C_1) + C_2\xi\psi_0^{(7)}(\xi) + (1+C_1)\psi_1^{(7)}(\xi),$$
$$\psi_2(0)=0,\ \psi_2'(0)=0,\ \psi_2''(0)=0,\ \psi_0'''(0)=0 \quad (21)$$
$$\psi_2(1)=0,\ \psi_2'(1)=0,\ \psi_2''(1)=0.$$

Using Eqns. (19), (20), and (21), the second order approximate solution for $p=1$ is:

$$\tilde{\psi}(\xi) = \psi_0(\xi) + \psi_1(\xi, C_1) + \psi_2(\xi, C_1, C_2) + O(\xi^{14}).$$

Following the procedure described in Section 2 for determination of optimal values of $C's$ on the domain between $a=0$ and $b=1$, we find the following values:

$$C_1 = -1.356239941 \ \& \ C_2 = -0.1269056771.$$

By considering these values, the approximate solution becomes:

$$\tilde{\psi}(\xi) = 1 - \xi^2/2 - \xi^3/3 - 0.12500000\,\xi^4 - 0.03333332\,\xi^5 - 0.00694445\,\xi^6 -$$
$$0.00119047\,\xi^7 - 0.00017361\xi^8 - 0.00002204\xi^9 - 2.4802\times10^{-6}\,\xi^{10} \quad (22)$$
$$-2.5052\times10^{-7}\,\xi^{11} - 2.2571\times10^{-8}\,\xi^{12} - 2.4473\times10^{-9}\,\xi^{13} + O(\xi^{14}).$$

The second order approximate solution using the auxiliary function $h(p) = p(C_1 + C_2\xi),$ is given by:

$$\tilde{\psi}(\xi) = \psi_0(\xi) + \psi_1(\xi, C_1) + \psi_2(\xi, C_1, C_2) + O(\xi^{14}).$$

Using the same procedure for C's we get,

$$C_1 = -1.00037658 \ \& \ C_2 = 0.00073941.$$

By considering these values, the approximate solution becomes:

$$\begin{aligned}\tilde{\psi}(\xi) &= 1-\xi^2/2-\xi^3/3-0.12500000\,\xi^4-0.03333333\,\xi^5-0.00694444\,\xi^6\\ &-0.00119048\,\xi^7-0.00017361\xi^8-0.00002205\xi^9-2.4802\times10^{-6}\xi^{10}\\ &-2.5052\times10^{-7}\xi^{11}-2.2964\times10^{-8}\xi^{12}-1.9265\times10^{-9}\xi^{13}+\mathrm{O}(\xi^{14}).\end{aligned} \quad (23)$$

Table 2 shows the error obtained by using the two types of auxiliary functions for different values of ξ in the domain of the problem. This table also shows comparison of the OHAM results with results of Adomian decomposition method [3] and the differential transform method [4].

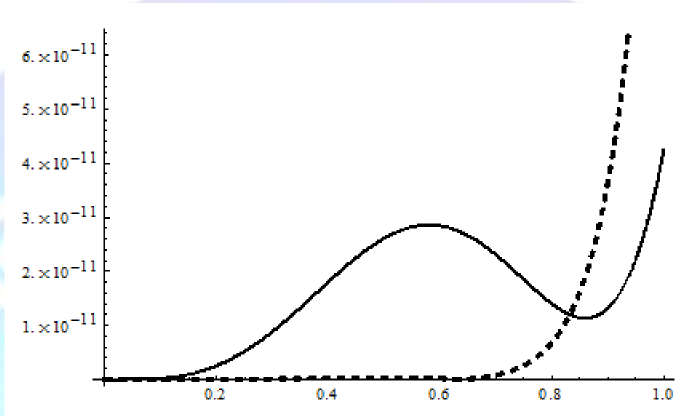
Table 2: Comparison of numerical results for problem 3.2

ξ	Absolute Error OHAM (22)	Absolute Error OHAM (23)	Absolute Error ADM	Absolute Error DTM
0.0	0.	0.	0	0
0.1	2.2138E-13	3.8858E-15	4.3972E-10	4.6585E-13
0.2	2.5473E-12	3.9746E-14	4.9251E-10	5.7126E-12
0.3	8.8516E-12	1.2534E-13	7.4067E-10	2.1299E-11
0.4	1.8052E-11	2.3348E-13	6.6537E-10	4.6995E-11
0.5	2.6125E-11	3.0387E-13	3.0059E-11	7.4307E-11
0.6	2.8426E-11	1.9751E-13	4.3591E-10	8.9219E-11
0.7	2.3050E-11	8.2212E-13	3.6735E-10	7.9767E-11
0.8	1.3979E-11	6.8465E-12	7.2753E-10	4.6686E-11
0.9	1.3217E-11	3.6504E-11	7.0036E-10	1.0960E-11
1.0	4.2523E-11	1.6089E-10	2.2191E-10	6.9252E-16

Error = Exact - Approximate

Figure 2 : Dashed - Absolute error curve for approximate solution (23)

Solid- Absolute error curve for approximate solution (22)



Example 3.3

Consider the following seventh order nonlinear boundary value problem:

$$\psi^{(7)}(\xi) = -\psi^2(\xi)Exp(\xi),\ 0 < \xi < 1,$$
$$\psi(0) = 1,\ \psi^{(1)}(0) = -1,\ \psi^{(2)}(0) = 1,\ \psi^{(3)}(0) = -1,$$
$$\psi(1) = Exp(-1),\ \psi^{(1)}(1) = -Exp(-1),\ \psi^{(2)}(1) = Exp(-1).$$

Exact solution of the problem is $\psi(\xi) = Exp(\xi).$

We use the usual auxiliary function $h(p) = pC_1 + p^2 C_2.$

Following the rest of procedure of OHAM, the following linear problems are obtained:

Zeroth order problem:

$$\begin{aligned} &\psi_0^{(7)}(\xi) = 0, \\ &\psi_0(0) = 1,\ \psi_0'(0) = -1,\ \psi_0''(0) = 1,\ \psi_0'''(0) = -1, \\ &\psi_0(1) = Exp(-1),\ \psi_0'(1) = -Exp(-1),\ \psi_0''(1) = Exp(-1). \end{aligned} \tag{24}$$

First order problem:

$$\psi_1^{(7)}(\xi, C_1) = (1+C_1)\psi_0^{(7)}(\xi) + C_1 Exp(\xi)\psi_0^2(\xi),$$
$$\psi_1(0)=0,\ \psi_1'(0)=0,\ \psi_1''(0)=0,\ \psi_1'''(0)=0, \quad (25)$$
$$\psi_1(1)=0,\ \psi_1'(1)=0, \psi_1''(1)=0.$$

Second order problem:

$$\psi_2^{(7)}(\xi, C_1, C_2) = (1+C_1)\psi_1^{(7)}(\xi) + C_2\psi_0^{(7)}(\xi) + \psi_0(\xi)Exp(\xi)(C_2\psi_0(\xi) + 2C_1\psi_1(\xi)),$$
$$\psi_2(0)=0,\ \psi_2'(0)=0,\ \psi_2''(0)=0,\ \psi_0'''(0)=0, \quad (26)$$
$$\psi_2(1)=0,\ \psi_2'(1)=0,\ \psi_0''(1)=0.$$

Using Eqns. (24), (25), and (26), the second order approximate solution for $p=1$ is:

$$\tilde{\psi}(\xi) = \psi_0(\xi) + \psi_1(\xi, C_1) + \psi_2(\xi, C_1, C_2) + \mathrm{O}(\xi^{14}).$$

Following the procedure described in Section 2 for determination of optimal values of $C's$ on the domain between $a=0$ and $b=1$, we find the following values:

$$C_1 = -0.70560359 \ \ \& \ \ C_2 = -0.08667069.$$

By considering these values, the approximate solution becomes:

$$\tilde{\psi}(\xi) = 1 - \xi + \xi^2/2 - \xi^3/6 + 0.04166666\xi^4 - 0.00833333\xi^5 + 0.00138889\xi^6$$
$$-0.00019841\xi^7 + 0.00002480\xi^8 - 2.7557\times10^{-6}\xi^9 + 2.7557\times10^{-7}\xi^{10} \quad (27)$$
$$-2.4967\times10^{-8}\xi^{11} + 1.9790\times10^{-9}\xi^{12} - 1.0598\times10^{-10}\xi^{13} + \mathrm{O}(\xi^{14}).$$

The second order approximate solution using the auxiliary function $h(p) = p(C_1 + C_2\xi),$ is given by:

$$\tilde{\psi}(\xi) = \psi_0(\xi) + \psi_1(\xi; C_1) + \psi_2(\xi; C_1, C_2) + \mathrm{O}(\xi^{14}).$$

Using the same procedure for C's we get,

$$C_1 = -0.99886718 \ \ \& \ \ C_2 = -0.00246486.$$

By considering these values, the approximate solution becomes:

$$\tilde{\psi}(\xi) = 1 - \xi + \xi^2/2 - \xi^3/6 + 0.04166667\xi^4 - 0.00833333\xi^5 + 0.00138889\xi^6$$
$$-0.00019841\xi^7 + 0.00002480\xi^8 - 2.7557\times10^{-6}\xi^9 + 2.7556\times10^{-7}\xi^{10} \quad (28)$$
$$-2.5049\times10^{-8}\xi^{11} + 2.0869\times10^{-9}\xi^{12} - 1.6012\times10^{-10}\xi^{13} + \mathrm{O}(\xi^{14}).$$

The second order approximate solution using the auxiliary function $h(p) = p(C_1 + C_2 Exp(\xi)),$ is given by:

$$\tilde{\psi}(\xi) = \psi_0(\xi) + \psi_1(\xi\,; C_1) + \psi_2(\xi\,; C_1, C_2) + O(\xi^{14}).$$

Using the same procedure for C's we get,

$$C_1 = -1.00022912 \;\; \& \;\; C_2 = 0.00034255.$$

By considering these values, the approximate solution becomes:

$$\begin{aligned}\tilde{\psi}(\xi) &= 1 - \xi + \xi^2/2 - \xi^3/6 + 0.04166667\xi^4 - 0.00833333\xi^5 + 0.00138889\xi^6 \\ &= 0.00019841\xi^7 + 0.00002480\xi^8 - 2.7557\times10^{-6}\xi^9 + 2.7557\times10^{-7}\xi^{10} \\ &-2.5052\times10^{-8}\xi^{11} + 2.0876\times10^{-9}\xi^{12} - 1.6052\times10^{-10}\xi^{13} + O(\xi^{14}).\end{aligned} \tag{29}$$

Numerical results of the approximate solutions (27), (28), (29) and the results obtained using Adomian decomposition method [3] and differential transform method [4] are displayed in table 3. These results for different values of ξ in the domain of the problem reveal that OHAM is more efficient than ADM and DTM in this case.

Table 3: Comparison of numerical results for problem 3.3

ξ	Absolute Error OHAM (27)	Absolute Error OHAM (28)	Absolute Error OHAM (29)	Absolute Error ADM	Absolute Error DTM
0.0	0.	0.	0.	0	0
0.1	5.7732E-15	1.8874E-15	1.1102E-16	1.5676E-9	3.0198E-14
0.2	5.7621E-14	2.0095E-14	0.000	1.6418E-9	3.6903E-13
0.3	1.7331E-13	6.2950E-14	5.5511E-16	4.9680E-9	1.3749E-12
0.4	3.0376E-13	1.1624E-13	8.8818E-16	1.5514E-9	3.0308E-12
0.5	3.7481E-13	1.5632E-13	1.8874E-15	1.5274E-9	4.7868E-12
0.6	3.4617E-13	1.6842E-13	9.8810E-15	2.4958E-9	5.7388E-12
0.7	2.3620E-13	1.9668E-13	7.5051E-14	1.3993E-8	5.1207E-12
0.8	1.0686E-13	5.3474E-13	4.7817E-13	2.5593E-9	2.9893E-12

0.9	2.9254E-14	2.4520E-12	2.4690E-12	5.4089E-9	6.9944E-13
1.0	4.5686E-14	1.0597E-11	1.0725E-11	1.1034E-9	1.1102E-16

Error = Exact – Approximate

Figure 3: Solid- Absolute error curve for approximate solution (27)

Dashed - Absolute error curve for approximate solution (28)

Solid Thick- Absolute error curve for approximate solution (29)

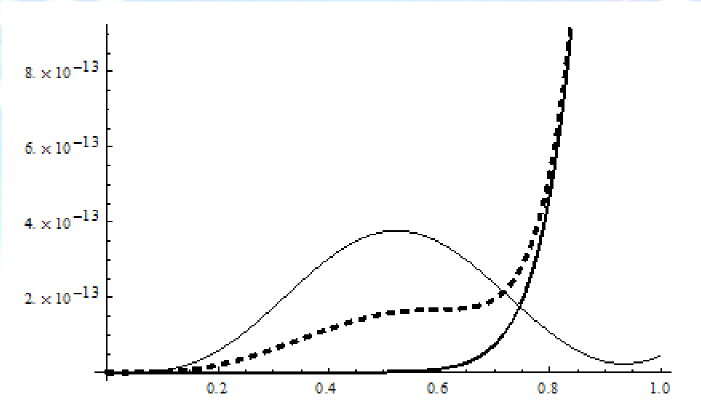


Figure 4: Dashed - Absolute error curve for approximate solution (28)

Solid- Absolute error curve for approximate solution (27)

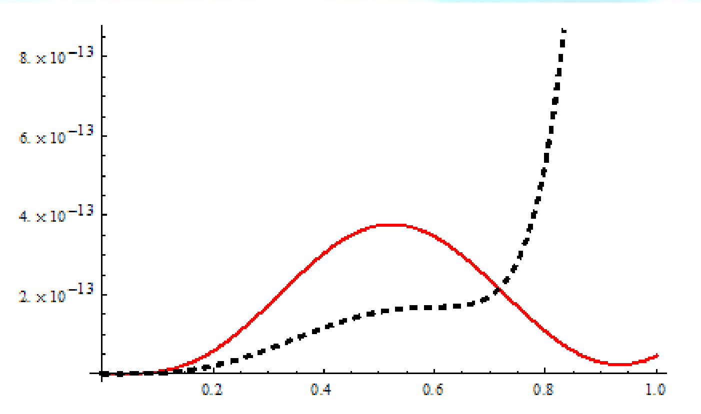


Figure 5: Dashed - Absolute error curve for approximate solution (29)

Solid- Absolute error curve for approximate solution (27)

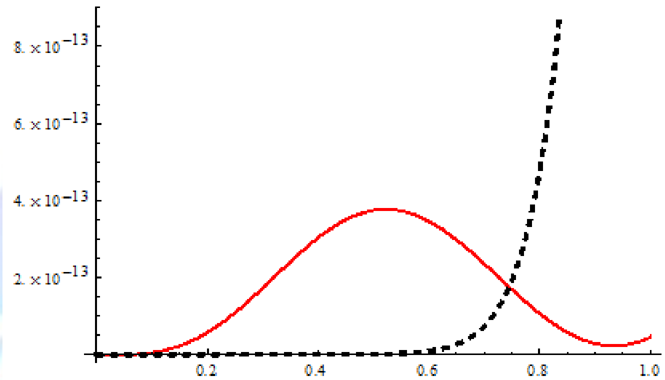
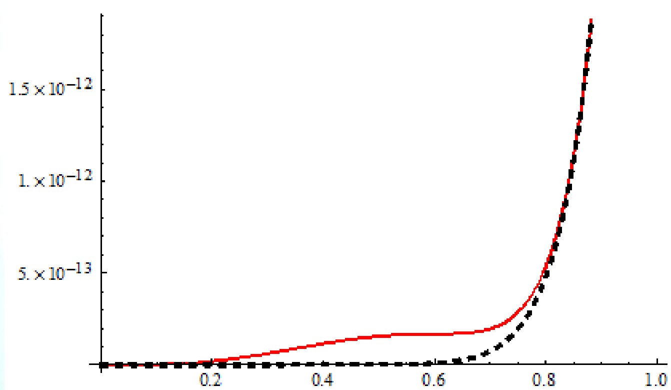


Figure 6: Dashed - Absolute error curve for approximate solution (29)

Solid- Absolute error curve for approximate solution (28)



4. Conclusion

In this work, the optimal homotopy asymptotic method with different auxiliary functions has been applied to obtain the numerical solutions of linear and nonlinear seventh order boundary value problems. Numerical results reveal higher degree accuracy for different types of auxiliary function as compared to the usual auxiliary function. This work shows that we should explore the beauty of OHAM by choosing variety of auxiliary functions and it is not necessary to stick to the usual auxiliary function $h(p)=PC_1+P^2C_2+\ldots$ all the times everywhere.

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