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ANALYSIS OF SOME METRIC SOLUTION WITH GEOMETRIC FORM

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Abstract. In the present paper we have studied the analysis of some metric solution with geometric form differs by its original rectangular, polar, cylindrical and spherical and other coordinates. By using the technique of Eigen value of characteristic equation of λ-tensor, Schwarzschild solution and others has been studied in metric tensor. It is assumed that, in section one contains a brief introduction to metric, spherical Reissner Nordstrom metric and Ricci flow. While in section two, defines the some metric solution and apply Christoffel symbol in the metric tensor. In the end; we are discussion the important role of metric tensor.

Keywords - Metric, Ricci solution, curvature, Christoffel symbols, Riemannian tensor etc.

2010 SUBJECT CLASSIFICATION: 83C15, 83C57, 53B50.

1. Introduction. Metric tensor is involved in theories of gravitational physics. Sometimes these models are defined under the conditions by a set of differential equations and governed by some rules for translating the mathematical results into physical world with meaningful statements. In general relativity our main motivation is to solve the Einstein's field equations. There are so many exact and non-exact solutions for these equations in the literature (c.f., [6]). In Einstein's theory of general relativity, the special metric solution discovered by karl Schwarzschild in 1916, describes the gravitational field outside a spherically symmetric, uncharged, non-rotating gravitational object such as a (non-rotating) star, planet, or black hole. The cosmological constant is assumed to equal zero. If we suppose the gravitational mass as sun, then the field outside the sun is called the some metric solution, given by the metric tensor. We define the arc length ds is obtained from $ds^2 = dx^2 + dy^2 + dz^2$. By transforming to general curvilinear coordinates in the space to be given by the metric form

1.1 $$ds^2 = \sum_{p=1}^{n}\sum_{q=1}^{n} g_{pq} dx^p dx^q$$

By the analysis of some metric solution with geometric form is $\Rightarrow ds^2 = g_{pq} dx^p dx^q$. The quantities g_{pq} are the components of a covariant tensor of rank 2 called the metric tensor. Further the cases of 2 and 3-dimension for metric solution are discussed, in which Gaussian curvature is calculated and shown its dependence on characteristic value of λ-tensor.

1.2 $(ds)^2 = (dr)^2 + r^2(d\theta)^2 + (dz)^2$

1.3 $(ds)^2 = r^2(d\theta)^2 + r^2 \sin^2\theta(d\phi)^2$

1.4 $(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2 \sin^2\theta(d\phi)^2$

1.5 $$ds^2 = \left(\frac{r^2}{r^2 - 2mr}\right)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 - \left(\frac{r^2}{r^2 - 2mr}\right)^{-1} dt^2$$

The corresponding solution for a charged, spherical, non-rotating body, the Reissner Nordström metric is

1.6 $$ds^2 = \left(\frac{r^2}{r^2 + e^2 - 2mr}\right)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 - \left(\frac{r^2}{r^2 + e^2 - 2mr}\right)^{-1} dt^2$$

In 1982, Hamilton [5] introduced the Ricci flow

1.7 $$\frac{\partial g_{pq}}{\partial t} = 2R_{pq}$$

to study compact three-manifolds with positive Ricci curvature and he call equation 1.7 as evolution equation. Hamilton proved many important and remarkable theorems for the Ricci flow, and laid the foundation for the program to approach the Poincare's conjecture and Thurston's geometrization conjecture via the Ricci flow. Further the idea was extended to Ricci solution by pulling back the solutions of Ricci flow along a λ-dependent diffeomorphism. The Ricci solution is a manifold (M, g_{ij}) whose metric tensor for a vector field ξ on it satisfy the equation

1.8 $$R_{pq} - \frac{1}{2}L_\xi g_{pq} = kg_{pq}$$

Here k is a constant and R_{pq} is the Ricci tensor for metric g_{pq}. The solution is gradient if $\xi = \nabla\phi$, for some function ϕ and steady if $k = 0$. If $k < 0$ the solution is called an expander, if $k > 0$ it is a shirker.

For four dimensional case Akbar and Woolger [3] have given a local $k = 0$ solution, named as metric solution. Further the Ricci solution for Lorentzian signature has been studied by Ali and Ahsan [2] and they have explored the case of Riesner-Nordstrom metric as a solution. The metric solution is obtained by deforming the original metric for a proper substitution of functions and vector fields, for which the new metric tensor satisfy the equation 1.8. The metric solution is given by the following equation (c.f., [2]) 1.9 $$ds^2 = -\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} dt^2 + dr^2 + (r^2 - 2mr)(d\theta^2 + \sin^2\theta d\phi^2)$$

Motivated by the all important role of Ricci solution in differential geometry and relativity, we have studied this concept for the space-time of general relativity. We have chosen the metric and studied its solution in detail. By using the 6-dimensional formalism, the characteristic values λ-tensor (i.e. $R_{AB} - \lambda g_{AB}$) has been given in this paper and an example of canonical form of the system is shown. Further the cases of 2 and 3-dimension for metric solution are discussed, in which Gaussian curvature is calculated and shown its dependence on characteristic value of λ-tensor.

2. Some Metric Solution and its Coordinates

2.1 $$[pq,k] = \frac{1}{2}\left[\frac{\partial g_{qk}}{\partial x^p} + \frac{\partial g_{pk}}{\partial x^q} - \frac{\partial g_{pq}}{\partial x^k}\right]$$

2.2 $$\begin{Bmatrix} k \\ pq \end{Bmatrix} = g_{kl}[pq,l]$$

The metric in cylindrical coordinates in equation 1.2 is

2.3 $$g = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = r^2$$

2.4 $$g_{11} = 1, \quad g_{22} = \frac{1}{r^2}, \quad g_{33} = 1$$

The Christiffel symbols of the first kind from equation 2.1 are

2.5 $$[22,1] = -r, \qquad [33,1] = [13,3] = [23,3] = 0$$

The Christiffel symbols of the second kind from equation 2.2 are

2.6 $$\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -r, \qquad \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 23 \end{Bmatrix} = 0$$

Again using equation 1.3, we have

2.7 $$g = \begin{vmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{vmatrix} = r^4\sin^2\theta \quad \text{or}$$

2.8 $g_{11} = \frac{1}{r^2}, \quad g_{22} = \frac{1}{r^2 \sin^2\theta}$

The Christiffel symbols of the first kind from equation 2.1 are

2.9 $[22,1] = r^2 \sin\theta\cos\theta, \qquad [12,2] = r^2 \sin\theta\cos\theta$

The Christiffel symbols of the second kind from equation 2.2 are

210 $\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -\sin\theta\cos\theta, \qquad \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = r^4 \sin\theta\cos\theta$

Further, also using equation 1.4, we have

2.11 $g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix} = r^4 \sin^2\theta$

2.12 $g_{11} = 1, \quad g_{22} = \frac{1}{r^2}, \quad g_{33} = \frac{1}{r^2 \sin^2\theta}$

The Christiffel symbols of the first kind from equation 2.1 are

2.13 $[22,1] = -r, \qquad [33,1] = -r\sin^2\theta, \qquad [13,3] = r\sin^2\theta, \qquad [23,3] = r^2 \sin\theta\cos\theta$

The Christiffel symbols of the second kind from equation 2.2 are

2.14 $\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -2, \qquad \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = -r\sin^2\theta, \qquad \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \frac{1}{r}, \qquad \begin{Bmatrix} 3 \\ 23 \end{Bmatrix} = \cot\theta$

Similarly**,** Schwarzschild metric equation 1.9 can be written in the following form

2.15 $ds^2 = dr^2 + (r^2 - 2mr)(d\theta^2 + \sin^2\theta d\phi^2) - \left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} dt^2$

The components of the potential for the gravitation or the metric tensor for metric solution 2.15 in spherical coordinates $x^\alpha \equiv (r, \theta, \phi, t)$ are given by

2.16 $$g_{pq}(\alpha)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^2 - 2mr & 0 & 0 \\ 0 & 0 & (r^2-2mr)\sin^2\theta & 0 \\ 0 & 0 & 0 & -\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}} \end{pmatrix}$$

or

2.17 $$g_{11}=1, g_{22}=(r^2-2mr), g_{33}=(r^2-2mr)\sin^2\theta, g_{44}=-\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}}$$

The kind of Christoffel symbols, can be calculated from the formula1.1 and 1.5 we have

2.18 $$\begin{Bmatrix} p \\ qk \end{Bmatrix} = g^{pl}[qk,l]$$

$$=\frac{1}{2}g^{pl}\left(\frac{\partial g_{pl}}{\partial x^k}-\frac{\partial g_{qk}}{\partial x^l}+\frac{\partial g_{kl}}{\partial x^q}\right)$$

Thus the non-zero components of the Christoffel symbols for metric 2.15, by using equation 2.17 are

2.19 $$\begin{Bmatrix} 1 \\ 22 \end{Bmatrix}=(m-r), \qquad \begin{Bmatrix} 1 \\ 33 \end{Bmatrix}=(m-r)\sin^2\theta$$

$$\begin{Bmatrix} 1 \\ 44 \end{Bmatrix}=\frac{\sqrt{2m}}{r^2-2mr}\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}}, \qquad \begin{Bmatrix} 2 \\ 12 \end{Bmatrix}=\begin{Bmatrix} 2 \\ 21 \end{Bmatrix}=\frac{r-m}{r^2-2mr}$$

$$\begin{Bmatrix} 2 \\ 33 \end{Bmatrix}=-\sin\theta\cos\theta, \qquad \begin{Bmatrix} 3 \\ 13 \end{Bmatrix}=\begin{Bmatrix} 3 \\ 31 \end{Bmatrix}=\frac{r-m}{r^2-2mr}$$

$$\begin{Bmatrix} 3 \\ 23 \end{Bmatrix}=\begin{Bmatrix} 3 \\ 32 \end{Bmatrix}=\cot\theta, \qquad \begin{Bmatrix} 4 \\ 14 \end{Bmatrix}=\begin{Bmatrix} 4 \\ 41 \end{Bmatrix}=\frac{\sqrt{2m}}{r^2-2mr}.$$

While Riemann tensor for the Schwarzschild solution 1.8 can be calculated from the formula [1]

2.20 $$R_{pqkl}=\frac{1}{2}\left(\frac{\partial^2 g_{pl}}{\partial x^q\partial x^k}+\frac{\partial^2 g_{qk}}{\partial x^p\partial x^l}-\frac{\partial^2 g_{pk}}{\partial x^q\partial x^l}-\frac{\partial^2 g_{ql}}{\partial x^p\partial x^k}\right)+g_{mn}\left(\begin{Bmatrix} m \\ qk \end{Bmatrix}\begin{Bmatrix} n \\ pl \end{Bmatrix}-\begin{Bmatrix} m \\ ql \end{Bmatrix}\begin{Bmatrix} n \\ pk \end{Bmatrix}\right)$$

And the non-zero components of Riemann tensor, by using equation 2.17 are

2.21 $R_{1212}=\frac{m^2}{r^2-2mr}, \qquad R_{1414}=\frac{2m}{(r^2-2mr)^2}\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}}[m+\sqrt{2(m-r)}]$

$$R_{2323}=-m^2\sin^2\theta, \qquad R_{2424}=\frac{-\sqrt{2m(m-r)}}{(r^2-2mr)}\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}}$$

$$R_{3131}=\frac{m^2\sin^2\theta}{r^2-2mr}, \qquad R_{3434}=\frac{-\sqrt{2}m(m-rs)\sin^2\theta}{(r^2-2mr)}\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}}$$

We now use the 6-dimensional formalism in the pseudo-Euclidean space $\Re^6$ by making the identification [4]

2.22 pq : 23 31 12 14 24 34

A: 1 2 3 4 5 6

We also make use of the identification as

2.23 $g_{pk}g_{ql}-g_{pl}g_{qk}=g_{pqkl}\rightarrow g_{AB}$

Where $A,B=1,2,3,4,5,6$ and g_{ij} are the components of the metric tensor at an arbitrary point (x^α) of the metric solution, whose metric is given by equation 2.15. The new metric tensor $g_{AB}(A,B=1,2,3,4,5,6)$ is symmetric and non-singular.

The non-zero components of the metric tensor $g_{AB}(A,B=1,2,3,4,5,6)$ for equation 2.15 in 6-dimensional formalism, by using formulation 2.23 are as

2.24 $g_{11}(x^\alpha)=(r^2-2mr)^2\sin^2\theta, \qquad g_{22}(x^\alpha)=(r^2-2mr)\sin^2\theta$

$$g_{33}(x^\alpha)=(r^2-2mr), \qquad g_{44}(x^\alpha)=\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}}$$

$$g_{55}(x^\alpha)=-(r^2-2mr)\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}}, \qquad g_{66}(x^\alpha)=-(r^2-2mr)\sin^2\theta\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}}.$$

Similarly, we can transform the components of the Riemann tensor as $R_{pqkl}\rightarrow R_{AB}$. Thus, for example R_{1212} can be written as R_{33} [using the identification 2.22]. The non-zero components of the tensor R_{AB} under the identification (2.23) are

2.25 $R_{11}(x^{\alpha}) = -m^2 \sin^2\theta,$ $\quad R_{22}(x^{\alpha}) = \dfrac{m^2 \sin^2\theta}{r^2 - 2mr}$

$$R_{33}(x^{\alpha}) = \frac{m^2}{r^2 - 2mr}, \qquad R_{44}(x^{\alpha}) = \frac{2m}{(r^2 - 2mr)^2}\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}}[m + \sqrt{2}(m - r)]$$

$$R_{55}(x^{\alpha}) = \frac{-\sqrt{2}m}{(r^2 - 2mr)^2}\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}}, \quad R_{66}(x^{\alpha}) = \frac{-\sqrt{2}m(m - r)\sin^2\theta}{(r^2 - 2mr)^2}\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}}.$$

Further we use all these values to find a canonical form of the $\lambda-$tensor $R_{AB} - \lambda g_{AB}$. Next, we will be interested in Eigen values for the metric solution 1.5 that is the solution of the characteristic equation $|R_{AB} - \lambda g_{AB}| = 0$. By using equations 2.24 and 2.23 easily, we calculate these Eigen values and those are given by

2.26 $\lambda_1(r) = \dfrac{m^2}{(r^2 - 2mr)^2},$ $\quad \lambda_2(r) = \dfrac{m^2}{(r^2 - 2mr)^2} = \lambda_3(r)$

$$\lambda_4(r) = \frac{-2m}{(r^2 - 2mr)^2}[m + \sqrt{2}(m - r)], \qquad \lambda_5(r) = \frac{-\sqrt{2}m(m - r)}{(r^2 - 2mr)^2} = \lambda_6(r),$$

$\lambda_i, p = 1,2,3,4,5,6$ are the solution of the character equation $|R_{AB} - \lambda g_{AB}| = 0$ which depend on m and r. In other words we can say that for $\lambda_i, p = 1,2,3,4,5,6$ [equation 2.26], the determinant of $\lambda-$ tensor $R_{AB} - \lambda g_{AB}$ is zero. Thus we can transform the system in canonical form for values of $\lambda_i, p = 1,2,3,4,5,6$ as

2.27 $$g_{A'B'} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$R_{A'B'} = \begin{pmatrix} \lambda_1(\mathrm{r}) & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2(\mathrm{r}) & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3(\mathrm{r}) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_4(\mathrm{r}) & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_5(\mathrm{r}) & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda_6(\mathrm{r}) \end{pmatrix}$$

Thus in our case the geometry determined by $\lambda-$tensor is of the type G_1 [(1)(1)(11)(11)] in Segre symbols. From equation 2.27, we note that even if mass $m=0$, the metric is flat.

Case I- $\theta=0$ or $\theta=\pi$

When taking $\theta=0$ OR $\theta=\pi$ that is $d\theta=0$ Schwarzschild metric, given by equation 2.15, reduces to the form

2.28 $$^*ds^2 = dr^2 - \left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}} dt^2$$

Now equation (2.28) is a 2-dimensional surface now. The metric tensor $*g$ in coordinates $x^\alpha \equiv (r,t)$ is given

2.29 $$g_{pq}(x^\beta) = \begin{bmatrix} 1 & 0 \\ 0 & -\left(\frac{\mathrm{r}^2-2mr}{r^2}\right)^{\sqrt{2}} \end{bmatrix}$$

here $p,q=1,4$. Thus the hyper-surface for $\theta=0$ or $\theta=\pi$ (i.e., $*H_0$ or $*H_\pi$) degenerates to two dimensional surface. The non-zero component of Riemann curvature tensor for equation 2.28 is unique and given by

2.30 $$*R_{1414}(x^\beta) = \frac{2m}{(r^2-2mr)^2}\left(\frac{r^2-2mr}{r^2}\right)^{\sqrt{2}}[m+\sqrt{2}(m-r)]$$

So the Gaussian curvature $*K$ for surface $*H_0$ or $*H_\pi$ is

2.31 $$*K(x^\beta) = \frac{2m}{r^2-2mr}[m+\sqrt{2}(m-r)].$$

Equations 2.26 and 2.31 show that curvature of the 2-dimensional surface of the metric is related to the Eigen value $\lambda_4(r)$.

Case II- $2m < r < \infty, 0 < \theta < \pi$ and $\phi = 0$

For this case, equation (2.15) reduces to

2.32 $$ds^2 = dr^2 + (r^2 - 2mr)d\theta^2 - \left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} dt^2$$

The metric tensor $**g_{pq}$ for equation (2.32) in coordinate $x^\gamma \equiv (r, \theta, t)$ is given by

2.33 $$**g_{pq}(x^\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (r^2 - 2mr) & 0 \\ 0 & 0 & -\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} \end{bmatrix}$$

The non-zero component of Riemann curvature tensor for equation 2.32 is as following

2.34 $$R_{1212}(x^\gamma) = \frac{m^2}{r^2 - 2mr},$$

$$R_{1414}(x^\gamma) = \frac{2m}{(r^2 - 2mr)^2}\left(\frac{r^2 - 2mr}{r^2}\right)^{\sqrt{2}} [m + \sqrt{2}(m - r)],$$

So for 3-dimensional space 2.32, the Gaussian curvature at each point $x^\gamma \equiv (r, \theta, t)$ is given by the following three physical quantities

2.35 $$**K_1(x^\gamma) = \frac{**R_{2424}(x^\gamma)}{|**g_{24}|} = \frac{-\sqrt{2}m}{(r^2 - 2mr)^2}$$

$$**K_2(x^\gamma) = \frac{**R_{1414}(x^\gamma)}{|**g_{14}|} = \frac{-2m}{(r^2 - 2mr)^2}[m + \sqrt{2}(m - r)],$$

$$**K_4(x^\gamma) = \frac{**R_{1212}(x^\gamma)}{|**g_{12}|} = \frac{m^2}{(r^2 - 2mr)^2}.$$

Here $**g_{24}$ denotes the sub-matrix of $**g_{pq}$ corresponding to $x^1 = r$. It is clear from equations 2.12 and 2.21 that are the curvature of the 3-dimensional space of metric solution can be expressed in terms of a $\lambda-$tensor which happens to be the solutions (Eigen-values) of the characteristic equation $|R_{AB} - \lambda g_{AB}| = 0$.

3. Discussion. The key point of this paper is the concept of Metric tensor which is the backbone of manifold and also observed that we worked out on analysis of some metric solution by using characteristic of $\lambda-$ tensor $R_{AB} - \lambda g_{AB}$ we have also discussed 2 and 3-dimensional cases with geometric form. Manifold are important role of dealing the extended of n-dimensional space whose shape and size are not fixed but major some particular areas that is, clouds, trees, brain, nervous system, respiratory system, snowflakes, mountains ranges, lighting, river and much, much more. We see that the metric tensor is of type $G_1[(1)(1)(11)(11)]$ in equation 2.27. Gaussian curvature differs with that of metric and also the dependence of curvature on Eigen values of $\lambda-$tensor $R_{AB} - \lambda g_{AB}$ is not similar. Thus the deformation in metric of space-time is cause for change in space.

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