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APPROXIMATE ANALYTICAL SOLUTION OF SEEPAGE OF GROUND WATER IN SOIL

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ABSTRACT

In this paper, a hybrid method which combines the Adomian decomposition method (ADM), the Laplace transform algorithm and the Pade approximant is introduced to solve the approximate analytical solution of a quadratic Riccati differential equation of fractional order, which occur when we solve seepage of ground water in soil; this method is very useful to solve nonlinear differential equation of fractional order. The results reveal that the method is very effective and convenient, and has a great improvement in the ADM truncated series solution which diverges rapidly as the applicable domain increases for solving nonlinear differential equation of fractional order.

Keywords: Nonlinear differential equation, Riccati differential equation, seepage, Adomian decomposition method (ADM).

1 INTRODUCTION

Mathematical modeling of many physical systems leads to nonlinear ordinary differential equations. An effective method is required to analyze the mathematical modeling, which provides solutions conforming to physical reality, i.e, the real world of physics. Therefore, we must be able to solve nonlinear ordinary differential equations, in space and time, which may be strongly nonlinear. Common analytic procedures linearized the system or assume that nonlinearities are relative insignificant. Such procedures change the actual problem to make it attractable by the conventional methods. In short, the physical problem is transformed into a purely mathematical one for which the solution is readily available. This change, sometimes seriously, the solution. Generally the numerical methods such as Rungs Kutta method are based on the discretization techniques, and they only permit us to calculate the approximate solution for some values of time and space variables, which cause us to overlook some important phenomena such as chaos and bifurcations, because generally nonlinear dynamic systems exhibit some delicate structures in very small time and space intervals. Also, the numerical methods require computer-intensive calculations. The ability to solve nonlinear equations by an analytic method is important because linearization change the problem, perturbation is only reasonable when nonlinear effects are very small, and the numerical methods needs a substantial amount of computation but only get limited information. Since the beginning of the 1980s, Adomian's has presented and developed a so-called Decomposition Method for solving linear and nonlinear problems such as ordinary differential equations. Adomian proposed a decomposition method for solving frontier problems of physics. The Adomian decomposition method has been widely applied to solve ordinary differential equations(ODE), partial differential equations (PDE), stochastic differential equations and integral equations, linear or nonlinear. Adomian's Decomposition Method (ADM) consist of splitting the given solution into linear and nonlinear parts, inverting the highest-order derivative operator contained in the linear operator on both sides, Identifying the initial and/or boundary conditions and the terms involving the independent variables alone as initial approximation, decomposition the unknown function into series whose components are to be determined, decomposing the nonlinear function in terms of special polynomials called Adomian's polynomials, and finding the successive terms of the series solution by recurrent relation using Adomian's polynomials. ADM is quantitative rather than quantitative, analytic, requiring neither linearization nor perturbations, and continuous with no resort to discretization and consequent computer-intensive calculations.ADM is are relatively new approach to provide analytical approximation to linear and non-linea problems, and it is particularly valuable as a tool for Scientists and applied mathematicians, because it provides immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solution to both linear and non-linear differential equations without linearization or discretization. Over the past few years, many new alternative to the use of traditional methods for the numerical solution of differential equations have been proposed.

Khuri proposed a numerical Laplace decomposition algorithm to solve a class of nonlinear differential equations Yusufoglu solved the Duffing equation by Laplace decomposition algorithm. The ADM has been proved to be an efficient method, but somehow has a drawback that the solution given by ADM is essentially a truncated series solution. The ADM solution coincides with the Taylor expansion at the origin in the initial value problem and diverges rapidly as the applicable domain increases. Nevertheless, the Pade approximant can be applied to improve the convergent of the truncated series.

The nonlinear Riccati differential equation is important in the optimal control systems and has been investigated by plenty of effective methods for years . Desai solved the quadratic Riccati differential equation by homotopy perturbation metho, Abbasbandy solved Iterated He's homotopy perturbation method,and He's variational iteration method with Adomian polynomial , respectively.

In order to overcome the drawbacks of the ADM solution, a hybrid method of the Laplace Adomian decomposition method (LADM) combined with Pade approximant, named the LADM-Pade approximant technique, is introduced.

In this paper, we present a numerical and analytical solution for the Riccati differential equation arising in seepage ground water soil.

2The LADM-Pade approximant technique

Consider the nonlinear Riccati differential equation of the form

$$\frac{dx}{dh} = P(h) + Q(h)x + R(h)x^2 \qquad (1)$$

$$x(0) = \alpha \qquad (2)$$

The nonlinear operator is $f(x) = x^2$,where $P(h), Q(h), \text{and } R(h)$ are known functions. The initial condition shows a is real constant. If $R(h) = 0$, the Riccati equation would be linear. First, the LADM applies the Laplace transform (denoted throughout this paper by L) to both the side the equation (1)

$$L\left[\frac{dx}{dh}\right] = L[P(h)] + L[Q(h)x] + L\left[R(h)x^2\right] \quad (3)$$

Using the property of the Laplace transform, we have

$$sL[x] - x(0) = L[P(h)] + L[Q(h)x] + L[R(h)x^2] \quad (4)$$

From equation (7)

$$sL[x] - \alpha = L[P(h)] + L[Q(h)x] + L[R(h)x^2] \quad (5)$$

The LADM assumes the solution as an infinite series

$$x = \sum_{n=0}^{\infty} x_n \quad (6)$$

Where the terms x_n are computed recursively. The nonlinear operator $f(x) = x^2$ is decomposed as

$$f(x) = x^2 = \sum_{n=0}^{\infty} A_n \quad (7)$$

Where $A_n = A_n(x_0, x_1, x_2, ..., x_n)$ are the so-called Adomian polynomials [1] given as

$$A_0 = f(x_0),$$

$$A_1 = x_1 f^{(1)}(x_0),$$

$$A_2 = x_2 f^{(1)}(x_0) + \frac{1}{2!} x_1^2 f^{(2)}(x_0),$$

$$A_3 = x_3 f^{(1)}(x_0) + x_1 x_2 f^{(2)}(x_0) + \frac{1}{3!} x_1^3 f^{(3)}(x_0),$$

$$A_4 = x_4 f^{(1)}(x_0) + \left[\frac{1}{2!} x_2^2 + x_1 x_3\right] f^{(2)}(x_0) + \frac{1}{3!} x_1^2 x_2 f^{(3)}(x_0) + \frac{1}{4!} x_1^4 f^{(4)}(x_0),$$

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$$A_n = \sum_{v=1}^{n} c(v,n) f^{(v)}(x_0) \quad (8)$$

The first index of $c(v,n)$ is the order of derivatives from 1 to n, and the second is the order of the Adomian polynomial. It is to be noted that the sum of the subscripts in each term of A_n are equal to n in this scheme. The $c(v,n)$ are products(or sums of products) of v components of x whose subscripts sum to n, divided by the factorial of the number of repeated subscripts. Thus $c(1,3)$ can only be x_3 where $c(2,3)$ will be x_1x_2 and $c(3,3)$ is $\frac{1}{3!}x_1^3$.

For , $f(x) = x^2$ the first 12 polynomials are

$$A_0 = x_0^2,$$

$$A_1 = 2x_0x_1,$$

$$A_2 = 2x_0x_2 + x_1^2,$$

$$A_3 = 2x_0x_3 + 2x_1x_2,$$

$$A_4 = 2x_0x_4 + 2x_1x_3 + x_2^2,$$

$$A_5 = 2x_0x_5 + 2x_1x_4 + 2x_2x_3,$$

$$A_6 = 2x_0x_6 + 2x_1x_5 + 2x_2x_4 + x_3^2,$$

$$A_7 = 2x_0x_7 + 2x_1x_6 + 2x_2x_5 + 2x_3x_4,$$

$$A_8 = 2x_0x_8 + 2x_1x_7 + 2x_2x_6 + 2x_3x_5 + x_4^2,$$

$$A_9 = 2x_0x_9 + 2x_1x_8 + 2x_2x_7 + 2x_3x_6 + 2x_4x_5,$$

$$A_{10} = 2x_0x_{10} + 2x_1x_9 + 2x_2x_8 + 2x_3x_7 + 2x_4x_6 + x_5^2,$$

$$A_{11} = 2x_0x_{11} + 2x_1x_{10} + 2x_2x_9 + 2x_3x_8 + 2x_4x_7 + 2x_5x_6. \quad (9)$$

Substituting equation (6) and (7) into (9), the result is

$$\mathrm{L}\left[\sum_{n=0}^{\infty} x_n\right] = \frac{\alpha}{s} + \frac{1}{s}\mathrm{L}[\mathrm{P(h)}] + \frac{1}{s}\mathrm{L}\left[\mathrm{Q(h)}\sum_{n=0}^{\infty} x_n\right] + \frac{1}{s}\mathrm{L}\left[\mathrm{R(h)}\sum_{n=0}^{\infty} A_n\right] \quad (10)$$

$$L[x_0] = \frac{\alpha}{s} + \frac{1}{s}L[R(h)], \quad (11)$$

$$L[x_1] = \frac{1}{s}L[Q(h)x_0] + \frac{1}{s}L[R(h)A_0], \quad (12)$$

$$L[x_2] = \frac{1}{s}L[Q(h)x_1] + \frac{1}{s}L[R(h)A_1], \quad (13)$$

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$$L[x_n] = \frac{1}{s} L\left[Q(h) x_{n-1}\right] + \frac{1}{s} L\left[R(h) A_{n-1}\right], \qquad n \geq 1 \qquad (14)$$

By applying the inverse Laplace transform to the equation (11), the value x_0 is obtained. Substituting this value x_0 to the (9), the first Adomian polynomial A_0 is obtained. Then substituting x_0 and A_0 to equation (12). Evaluating the Laplace transform of the quantities on the right-hand side of the equation (12) and then apply the inverse Laplace transform, the value of x_1 can be obtained. The other terms $x_2, x_3, x_4, \ldots,$ can be computed recursively in a similar calculation by equation (14).

By using LADM, a power series solution x_n is essentially a truncated series solution. The LADM solution coincides with the Maclaurin series of the exact solution in the initial value case and diverges rapidly as the applicable domain increases. However, the Pade approximant extends the domain of the truncated series solution to obtain better accuracy and convergence. The Pade approximant approximates a function by the ratio of two polynomials. The [L/M] approximant to a formal power series B(h) is given by

$$\left[\frac{L}{M}\right] = \frac{P_L(h)}{Q_M(h)} = \frac{a_0 + a_1 h + a_2 h^2 + \ldots + a_L h^L}{b_0 + b_1 h + b_2 h^2 + \ldots + b_M h^M}, \qquad (15)$$

$$B(h) = \sum_{j=0}^{\infty} b_j h^j, \qquad (16)$$

Where $P_L(h)$ is a polynomial of degree at most L and $Q_M(h)$ is a polynomial of degree at most M. It can be observed in equation (20) that there are L+1 numerator coefficients and M+1 denominator coefficients. If the denominator coefficient b_0 is defined to be 1,then the L+1 numerator coefficients will still remain unchanged, but the number of denominator coefficient will reduced to M. Thus, there are L+M+1 unknown coefficients, and the number [L/M] should fit the power series B (h) through the orders $(1, h, h^2, h^3, \ldots, h^{L+M})$. Therefore, the [L/M] Pade approximant can represent a power series $\sum_{j=0}^{\infty} b_j h^j$ by a rational fraction of polynomials $\frac{P_L(h)}{Q_M(h)}$ with an error $O(h^{L+M+1})$

$$\sum_{j=0}^{\infty} b_j h^j = \frac{P_L(h)}{Q_M(h)} + O(h^{L+M+1}) \qquad (17)$$

3 STATEMENT OF PROBLEM

To understand this problem, here it is assumed that water from a head reservoir, flow into adjacent soil, which stands on inclined bedrock and exhibits heterogeneity in the vertical direction. After seeping over considerable distance, it falls into a tail reservoir. To examine here the nature of the free surface of flow in a vertical plane when the seepage face is neglected. We choose a horizontal line at the bottom of the tail reservoir as the X-axis, a vertical line besides it as Z-axis. The inclined boundary is the line z = -mx, where $m = \tan\alpha$, $\left(0 < \alpha < \frac{\pi}{2}\right)$ is the slope of the inclined bedrock (Figure 1).

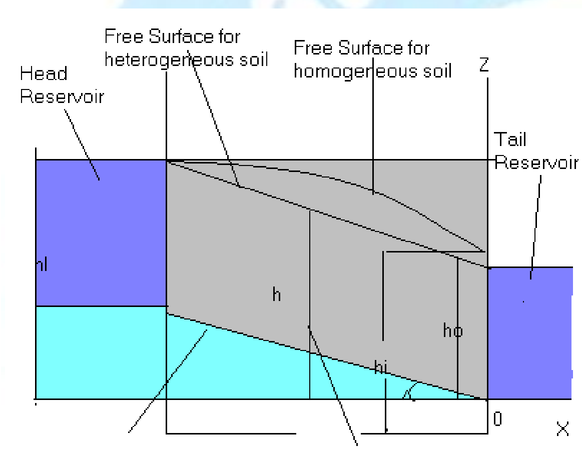

Figure 1: Seepage over inclined bedrock

3 MATHEMATICAL FORMULATION OF THE PROBLEM

According to Darcy's law, the seepage velocity is given by

$$u = -k(z)\frac{dh}{dx} \tag{18}$$

Where h is the piezometric head and k (z) is the seepage coefficient of the porous medium, which varies with z linearly.
For definiteness we assume that

$$k(z) = k_0(1 - bz) \tag{19}$$

Where k_0 and b are non-zero constants and b>0, (Verma 1965)

When such flow of groundwater takes place over considerable distance, the analysis is based on hydraulic theory. In the hydraulic theory, the h is equal to the height of the free surface (neglecting the atmospheric pressure) and the flow elements depend on x alone. The flow rate q_x is given by

$$q_x = -\frac{dh}{dx}\int_0^h k(z)\,dz \tag{20}$$

Where z = 0 is the foot and z = h is the top of the vertical section at a distance x for which q_x is measured; $\frac{dh}{dx}$ is independent of z.

The equation of continuity is

$$\frac{dq_x}{dx} = 0 \tag{21}$$

$$q_x = \text{constant} = q \qquad (\text{say}) \tag{22}$$

From equation (19), (20), and (22)

We obtain

$$q = -k_0 \frac{dh}{dx}\int_{-mx}^{h} (1 - bz)\,dz \tag{23}$$

Performing the integration and on rearrangement we obtain,

$$\frac{dx}{dh} = P(h) + Qx + Rx^2 \tag{24}$$

Where

$$P(h) = -\frac{k_0}{q}\left(h - \frac{b}{2}h^2\right) = 1 \qquad (25)$$

$$Q = -\frac{k_0 m}{q} \quad = 2 \qquad (26)$$

$$R = -\frac{k_0 b m^2}{2q} \quad = -1 \qquad (27)$$

The equation (24) is the generalized Riccati's equation.

Hence equation (24) becomes

$$\frac{dx}{dh} = 1 + 2x(h) - x^2(h),\ h > 0 \qquad (28)$$

Subject to initial condition x (0) = 0 (29)

4 EXACT AND HOMOTOPY PERTURBATION SOLUTION

The exact solution, is

$$x(h) = 1 + \sqrt{2}\tanh\left(\sqrt{2}h + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right) \qquad (30)$$

And we can observe that, as $h \to \infty, x(h) \to 1+\sqrt{2} = 2.4142$

First, the LADM applies the Laplace transform to both sides of equation (28)

$$L[x^{'}] = L[1] + 2L[x] - L[x^2] \tag{31}$$

Using the property of the Laplace transform, we have

$$sL[x] - x(0) = \frac{1}{s} + 2L[x] - L[x^2] \tag{32}$$

Using the initial condition

$$sL[x] - 2L[x] = \frac{1}{s} - L[x^2] \tag{33}$$

$$L[x] = \frac{1}{s(s-2)} - \frac{1}{(s-2)} L[x^2] \tag{34}$$

Substituting equation (6) and (7) into (34), the result is

$$L\left[\sum_{n=0}^{\infty} x_n\right] = \frac{1}{s(s-2)} - \frac{1}{(s-2)} L\left[\sum_{n=0}^{\infty} A_n\right] \tag{35}$$

Matching both sides of equation (35) we have

$$L[x_0] = \frac{1}{s(s-2)}, \tag{36}$$

$$L[x_1] = \frac{-1}{s-2} L[A_0], \tag{37}$$

$$L[x_2] = \frac{-1}{s-2} L[A_1], \tag{38}$$

$$L[x_n] = \frac{-1}{s-2} L[A_{n-1}], \ \mathrm{n} \geq 1 \tag{39}$$

Applying the inverse Laplace transform to equation (36), we have

$$x_0 = \frac{-1}{2} + \frac{1}{2}e^{2h} \tag{40}$$

Substituting this value of x_0 into equation (9) and (37) gives

$$L[x_1] = -\frac{1}{s-2}L[x_0^2] \tag{41}$$

$$= -\frac{1}{(s-2)}L\left[\left(-\frac{1}{2}+\frac{1}{2}e^{2h}\right)^2\right]$$

$$= -\frac{1}{(s-2)}L\left[\frac{1}{4}-\frac{1}{2}e^{2h}+\frac{1}{4}e^{4h}\right]$$

$$L[x_1] = -\frac{1}{(s-2)}\left[\frac{1}{4s}-\frac{1}{2(s-2)}+\frac{1}{4(s-4)}\right]$$

Hence

$$x_1 = \frac{1}{8}+\frac{1}{2}he^{2h}-\frac{1}{8}e^{4h}, \tag{42}$$

Now

$$L[x_2] = -\frac{1}{(s-2)}L[2y_0y_1]$$

$$= -\frac{1}{(s-2)}L\left[2\left(-\frac{1}{2}+\frac{1}{2}e^{2h}\right)\left(\frac{1}{8}+\frac{1}{2}he^{2h}-\frac{1}{8}e^{4h}\right)\right]$$

$$= -\frac{1}{(s-2)}L\left[-\frac{1}{8}+\frac{1}{8}e^{2h}-\frac{1}{2}he^{2h}+\frac{1}{8}e^{4h}+\frac{1}{2}he^{4h}-\frac{1}{8}e^{6h}\right]$$

$$= -\frac{1}{(s-2)}\left[-\frac{1}{8s}+\frac{1}{8(s-2)}-\frac{1}{2(s-2)^2}+\frac{1}{8(s-4)}+\frac{1}{2(s-4)^2}-\frac{1}{8(s-6)}\right]$$

Hence

$$x_2 = -\frac{1}{16} - \frac{1}{32}e^{2h} - \frac{1}{8}he^{2h} + \frac{1}{4}h^2e^{2h} + \frac{1}{16}e^{4h} - \frac{1}{4}he^{4h} + \frac{1}{32}e^{6h}, \qquad (43)$$

The other terms $x_3, x_4, \ldots,$ can be computed recursively in a similar calculation by equation (39). The first 12 terms of the LADM solution are

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$$x_{12} = -\frac{52{,}003}{8{,}388{,}608} - \frac{81{,}719}{8{,}388{,}608}e^{2h} + \frac{7429}{524{,}288}he^{2h} - \frac{7429}{2{,}097{,}152}e^{4h} - \frac{3553}{524{,}288}h^2e^{2h} - \ldots \quad (44)$$

Therefore, the truncated series solution obtained from LADM is

$$x(h) = x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12}$$

$$= -\frac{3{,}499{,}153}{8{,}388{,}608} + \frac{4{,}031977}{8{,}388{,}608}e^{2h} + \frac{428{,}739}{1{,}048{,}576}he^{2h} - \frac{178{,}375}{2{,}097{,}152}e^{4h} + \frac{85{,}473}{524{,}288}h^2e^{2h} - \ldots$$

$$+ \frac{122}{4{,}194{,}304}h^2e^{22h} + \frac{9}{16{,}777{,}216}e^{24h} - \frac{3}{2{,}097{,}152}te^{24h} + \frac{1}{33{,}554{,}432}e^{26h}$$

(45)

The Taylor expansion of $x(h)$ about h=0 of the LADM solution is

$$x(h) = h + h^2 + 0.333333h^3 - 0.333333h^4 - 0.466666h^5 + O(h^6) \qquad (45)$$

The Taylor expansion of $x(h)$ about h=0 of the exact solution is

$$x(h) = 0.999999h + 0.999999h^2 + 0.333333h^3 - 0.333333h^4 - 0.466666h^5 + O(h^6) \qquad (46)$$

It shows both the Taylor expansions at h=0 of the LADM solution and exact solution coincide very well. In order to improve the LADM solution, the LADM-Pade approximant technique is introduced. According to equation (11) and (22), it is known that there exists a [L/M] Pade approximant which satisfies:

$$\sum_{n=0}^{\infty} x_n = \frac{P_L(h)}{Q_M(t)} + O\left(h^{L+M+1}\right) = \left[\frac{L}{M}\right] + O(h^{26}) \qquad (47)$$

The Pade approximant of the $\left[\frac{13}{12}\right]$ truncated series obtained from the LADM solution in equation (45) is found to be

$$\left[\frac{13}{12}\right] = \frac{h + 0.04h^2 + 0.2933333h^3 + ... + 3.157154347\times 10^{-10} h^{12} + 8.09526755\times 10^{-12} h^{13}}{1 - 0.96h + 0.92h^2 - ... - 4.420016086\times 10^{-9} h^{11} + 4.20953913\times 10^{-10} h^{12}} \qquad (48)$$

The absolute error can be defined as

Absolute error= $\left|(\text{exact solution}) - (\text{LADM} - \text{Pade approximant solution})\right|$.

Figure 2 illustrates the comparisons between the exact solution [14], the and the LADM –Pade approximant solution. Also the ADM and LADM solutions diverges rapidly after h=0.7 and 0.89, respectively. However, it illustrates that the LADM-Pade approximant solution demonstrates a very good convergence through the applicable domain.

Table 1 shows the absolute errors of the LADM

h	X(h)	
	LADM-pade approximant solution	Exact solution [14]
0	0	0
1	1.6894983916	1.6894983917
2	2.3577716533	2.3577716533
3	2.4108136861	2.4108136860
4	2.4140123825	2.4140123826
5	2.4142016594	2.4142016707
6	2.4142127085	2.4142128595
7	2.4142124514	2.4142135209
8	2.4141086330	2.4142135599
9	2.4141969185	2.4142135623
10	2.4141689847	2.4142135624
15	2.4134260559	2.4142135624
20	2.4119642426	2.4142135624
25	2.4129813400	2.4142135624
30	2.4204457663	2.4142135624

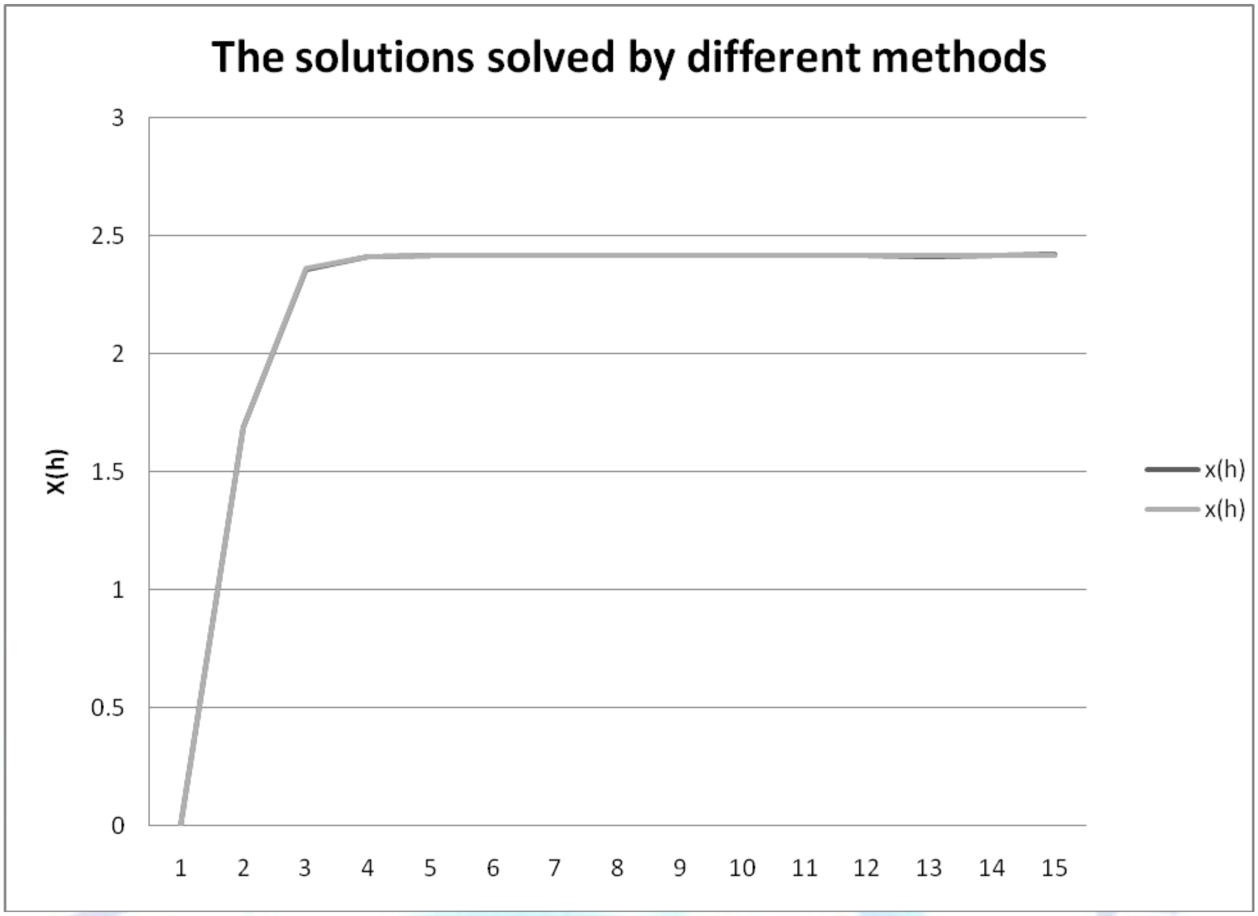

Figure 2

5 CONCLUSIONS

Both the ADM and LADM solutions are truncated Taylor series solutions, and diverge rapidly when $h \to 1$. The hybrid method of the LADM with Pade approximant developed in this paper has been proved to be an efficient and reliable method with good accuracy in solving the nonlinear differential Riccati equations. Moreover, the LADM-Pade approximant solution has been demonstrated not only the superiority of the accuracy and convergence over both the ADM and LADM solutions, but also extended the applicable domain to overcome their drawbacks.

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