GAUSSIAN NEIGHBOURHOOD PRIME LABELING OF SOME GRAPHS

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Abstract

A graph G on n vertices is said to have a neighbourhood prime labelling if there exists a labelling from the vertices of G to the first n natural numbers such that for each vertex in G with degree greater than one, the neighbourhood vertices have relatively prime labels. Gaussian integers are the complex numbers whose real and imaginary parts are both integers. We extend the neighbourhood prime labelling concept to Gaussian integers. Using the order on the Gaussian integers, we show that several families of trees admit Gaussian neighbourhood prime labelling.

Keywords: Neighbourhood Prime labelling, Gaussian integers

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1. Introduction

Graph labelling where the vertices are assigned values subject to certain conditions have many applications in Engineering and Science. For all terminology and notations in Graph theory, we follow [1] and for all terminology regarding graph labelling, we follow [2].The Prime labelling concept was introduced by Roger Entriger. A graph on n vertices is said to have a prime labelling if its vertices can be labelled with the first n natural numbers in such a way that any two adjacent vertices have relatively prime labels. In [4] Steven Klee, Hunter Lehmann and Andrew Park extend the notion of prime labelling to Gaussian integers. They define a spiral ordering on the Gaussian integers that allow us to linearly order the Gaussian integers. Consecutive Gaussian integers in the spiral ordering are relatively prime and Consecutive odd Gaussian integers in the spiral ordering are relatively prime. Steven Klee proved that the path graph, star graph, spider graph, n-centipede tree, double star tree and firecracker admits Gaussian prime labelling.

A graph G on n vertices is said to have a neighbourhood prime labelling if there exists a labelling from the vertices of G to the first n natural numbers such that for each vertex in G with degree greater than one, the neighbourhood vertices have relatively prime labels. In this Paper we extend the study of neighbourhood prime labelling to Gaussian integers. In section 2, we discuss the properties of spiral ordering in Gaussian integers. In section 3, we apply the properties of spiral ordering to prove the Gaussian neighbourhood prime labelling of some graphs.

2. Gaussian Neighbourhood Prime Labelling

All graphs in this paper are finite undirected graphs without loops or multiple edges. A vertex is a leaf or end vertex if it has degree one and that it is an internal node otherwise. We follow [4] for definition and information on the Gaussian integers. The Gaussian integers, denoted Z[i], are the complex numbers of the form a+bi, where a, b ϵZ and $i^2 = -1$. The norm of Gaussian integer a+bi, denoted by N(a+bi), is given by a^2+b^2. A Gaussian integer is even if it is divisible by 1+i and odd otherwise. A unit in the Gaussian integers is one of ±1,±i. An associate of a Gaussian integer α is u.α where u is a Gaussian unit. A Gaussian integer ρ is prime if its only divisors are ±1,±i,±ρ or ±ρi. The Gaussian integers α and β are relatively prime if their only common divisors are units in Z[i].

The Gaussian integers are not totally ordered. So to define the first n natural numbers we use the spiral ordering of the Gaussian integers introduced by Steven Klee in [4].

Definition 1 ([4]) : The spiral ordering of the Gaussian integers is a recursively defined ordering of the Gaussian integers. We denote the n^{th} Gaussian integer in the spiral ordering by γ_n. The ordering is defined beginning with γ_1=1 and continuing as:

$$\gamma_{n+1} = \begin{cases} \gamma_n + i, & if\ Re(\gamma_n) \equiv 1(mod2), Re(\gamma_n) > Im(\gamma_n) + 1 \\ \gamma_n - i, & if\ Im(\gamma_n) \equiv 0(mod2), Re(\gamma_n) \le Im(\gamma_n) + 1, Re(\gamma_n) > 1 \\ \gamma_n + 1, & if\ Im(\gamma_n) \equiv 1(mod2), Re(\gamma_n) < Im(\gamma_n) + 1 \\ \gamma_n + i, & if\ Im(\gamma_n) \equiv 0(mod2), Re(\gamma_n) = 1 \\ \gamma_n = i, & if\ Re(\gamma_n) \equiv 0(mod2), Re(\gamma_n) \ge Im(\gamma_n) + 1, Im(\gamma_n) > 0 \\ \gamma_n + 1, & if\ Re(\gamma_n) \equiv 0(mod2), Im(\gamma_n) = 0. \end{cases}$$

The first 10 Gaussian integers under this ordering are 1,1+i,2+i,2,3,3+i,3+2i,2+2i,1+2i,1+3i,............and $[\gamma_n]$ denote the set of the first n Gaussian integers in the spiral ordering. Here we exclude the imaginary axis to ensure that the spiral ordering excludes associates. Consecutive Gaussian integers in the spiral ordering are separated by a unit and therefore alternate parity, as in the usual ordering of N. Furthermore, odd integers with indices separated by a power of two are not guaranteed to be relatively prime to each other.

Now we define the neighbourhood prime labelling with Gaussian integers using the definition of the spiral ordering for Gaussian integers. For a vertex vϵV(G) ,the neighbourhood of v is the set of all vertices in G which are adjacent to v and is denoted by N(v).

Definition 2: Let G be a graph on n vertices. A Gaussian neighbourhood prime labelling of G is a bijection f :V(G) → $[\gamma_n]$ such that for each vertex vϵV(G) with deg(v) >1, {$f(u_i)$: u_i ϵN(v)} are relatively prime.

In [4] Steven Klee proved the following properties of Gaussian integers in spiral ordering.

(1) Let α be a Gaussian integer and u be a unit. Then α and α+u are relatively prime.

(2) Consecutive Gaussian integers in the spiral ordering are relatively prime.

(3) Let α be an odd Gaussian integer, let c be a positive integer, and let u be a unit. Then α and α + u.$(1+i)^c$ are relatively prime.

(4) Consecutive odd Gaussian integers in the spiral ordering are relatively prime.

(5) Let α be a Gaussian integer and let p be a prime Gaussian integer. Then α and α + p are relatively prime if and only if p does not divides α.

3. Gaussian neighbourhood prime labelling of some graphs

We now discuss the Gaussian neighbourhood prime labelling for some graphs using the properties of spiral ordering in the Gaussian integers.

Definition 3. The star graph $\mathbf{K_{1,n}}$ is the graph with n pendent edges incident with the vertex in $\mathbf{K_1}$

Theorem 1. The star graph $\mathbf{K_{1,n}}$ admits a Gaussian neighbourhood prime labelling.

Proof. The apex vertex in $K_{1,n}$ is adjacent to all the n vertices. Let v be the apex vertex and u_1, u_2,..............,u_n be the n pendent vertices. Label the apex vertex v with $\gamma_1 = 1$ and label the pendent vertices arbitrarily with the remaining Gaussian integers $\gamma_2, \gamma_3, \dots \dots \dots \dots \dots \dots \dots \dots ., \gamma_{n+1}$. The Gaussian integer $\gamma_1 = 1$ is relatively prime to all the Gaussian integers. Therefore the star graph $\mathbf{K_{1,n}}$ admits a Gaussian neighbourhood prime labelling.

Definition 4. A bistar graph is the graph obtained by joining the apex vertices of two copies of star graph $K_{1,n}$ and it is denoted by $B_{n,n}$. There are 2n+2 vertices in the bistar graph in which two vertices have degree n+1 and 2n vertices have degree one.

Theorem 2. The bistar graph $B_{n,n}$ admits a Gaussian neighbourhood prime labelling.

Proof. Let v and u be apex vertices of two copies of star graph $K_{1,n}$. The pendent vertices of each copies of the star graphs are v_1,v_2,v_3,..................,v_n and u_1,u_2,u_3,..................,u_n. The bistar graph is obtained by joining the vertices v and u. Label the vertex v with $\gamma_1 = 1$ and label the vertex u with $\gamma_2 = 1 + i$.Then label the remaining pendent vertices with consecutive Gaussian integers $\gamma_3, \gamma_4, \dots \dots \dots \dots \dots \dots \dots \dots ., \gamma_{n+2.}$.The Gaussian integers $\gamma_1 = 1$ and $\gamma_2 = 1 + i$ are relatively prime to all the Gaussian integers. Therefore the bistar graph $B_{n,n}$ admits a Gaussian neighbourhood prime labelling.

Definition 5. A double star tree is a graph consisting of the union of two stars $K_{1,n}$ and $K_{1,m}$ together with an edge joining their apex vertices. A double star has exactly two vertices that are not leaves.

Theorem 3. Any double star tree admits a Gaussian neighbourhood prime labelling.

Proof. Let G be a double star tree which is the union of two stars $K_{1,n}$ and $K_{1,m}$ ($n \le m$).Let v_1 and v_2 be the apex vertices of $K_{1,n}$ and $K_{1,m}$.Label the vertex v_2 with γ_1 =1 and v_1 with γ_2 = 1+i. The remaining m vertices in $K_{1,n}$ are labelled with the remaining n odd Gaussian integers and remaining m vertices in $K_{1,m}$ are labelled with the remaining Gaussian integers. The Gaussian integer 1+i is relatively prime to all the odd Gaussian integers and the Gaussian integer 1 is relatively prime to all the Gaussian integers. Therefore G admits a Gaussian neighbourhood prime labelling.

Definition 7. For each point v of a graph G, take a new point v'. Join v' to all points of G adjacent to v. The graph S(G) thus obtained is called the splitting graph of G.

Theorem. The splitting graph of star graph admits Gaussian neighbourhood prime labelling.

Proof. Let v be the apex vertex and $v_1,v_2,v_3,\ldots\ldots,v_n$ be the pendent vertices of the star graph $K_{1,n}$. Let $S(K_{1,n})$ be the splitting graph of $K_{1,n}$ .The newly added vertices in $S(K_{1,n})$ are $v_1', v_2', v_3', \ldots\ldots\ldots\ldots\ldots\ldots., v_n'$ and v' where v is connected to an edge with $v_1', v_2', v_3', \ldots\ldots\ldots\ldots\ldots\ldots., v_n'$ and v' is connected to an edge with $v_1,v_2,v_3,\ldots\ldots,v_n$.

We define a function L: $V(S(K_{1,n})) \rightarrow [\gamma_{2n+2}]$ as

$L(v) = 1$

$L(v') = 1+i$

$L(v_i) = \gamma_{2i-1}$, $1 \le i \le n$

$L(v_i') = \gamma_{2i}$, $1 \le i \le n$.

Clearly L is a bijection. The Gaussian integer 1+i is relatively prime to all odd Gaussian integers and 1 is relatively prime to all Gaussian integers. So the labelling is a Gaussian neighbourhood prime labelling.

Definition 6. A friendship graph or Dutch windmill graph F_n is a planar undirected graph with 2n+1 vertices and 3n edges. The friendship graph F_n can be constructed by joining n copies of the cycle graph C_3 with a common vertex.

Theorem 4. The friendship graph F_n admits Gaussian neighbourhood prime labelling.

Proof. Let v be the apex vertex of the friendship graph F_n. Let v_1 and v_2 be the other two vertices of the first copy of cycle C_3 and v_3 and v_4 are the other two vertices of second copy of cycle C_3 and so on. Finally $v_{2n-1}$and v_{2n} are other two vertices of n^{th} copy of cycle C_3.Label the vertex v with $\gamma_1 = 1$ .Then label the vertices $v_1,v_2,v_3,\ldots\ldots,v_{2n}$ with the consecutive Gaussian integers $\gamma_2,\gamma_3,\gamma_4,\ldots\ldots,\gamma_{2n+1}$. The neighbourhood vertices of every vertex other than apex vertex in the friendship graph contain the apex vertex. So the labelling is Gaussian neighbourhood prime.

Definition 7. A path graph $\mathbf{P_n}$ is a graph whose vertices can be listed in the order $v_1,v_2,v_3,\ldots\ldots,v_n$ such that the edges are $\{v_i,v_{i+1}\}$ where i =1,2,..................,n-1.In a path there are at most two vertices of degree 1 and all other vertices have degree 2.

Theorem 5. The path graph P_r admits a Gaussian neighbourhood prime labelling.

Proof. Let $v_1, v_2, v_3, \ldots, v_r$ are the vertices of path graph P_r. Consider the labelling $f: V(P_r) \rightarrow [\gamma_n]$ as follows.

Case(1) : If r is even

$$f(v_{2i-1}) = \gamma_{\left(\frac{r}{2}+i\right)}, 1 \leq i \leq \frac{r}{2}$$

$$f(v_{2i}) = \gamma_i, 1 \leq i \leq \frac{r}{2}$$

Case(2) :If r is odd

$$f(v_{2i-1}) = \gamma_{\left(\frac{r-1}{2}+i\right)}, 1 \leq i \leq \frac{r+1}{2}$$

$$f(v_{2i}) = \gamma_i, 1 \leq i \leq \frac{r-1}{2}.$$

The above labelling shows that adjacent vertices of all the vertices in the path P_r are consecutive Gaussian integers and the Gaussian integers in the spiral ordering are relatively prime. Therefore this is a Gaussian neighbourhood prime labelling for the path P_r.

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