On The Quotient Graph in The Cayley Graphs of Solvable Groups

Susan Ray Joseph,Department of Mathematics,M. A.College,Kothamangalam,

Mathew Varkey TK ,Department of Mathematics,TKM College of Engineeing,Kollam,

Abstract: The Cayley graphs of solvable groups has been studied by many . In this paper we prove that Quotient graph in the Cayley graph of a solvable group is normal.

Key words: Solvable groups ,Cayley graphs ,Quotient graph ,Normal Cayley graphs

1.Priliminaries:

For all the basic definitions and references other than the ones given in this paper refer to references [1],[2],[3],[4],[6],[7] and [8]

Definition:1.1: A group G is said to be solvable if it has a composition series $\{H_i\}$ such that all factor groups H_{i+1}/ H_i are abelian.

Definition1.2.: Cayley digraph of a group G with generating set S denoted by Cay(G,S) is defined as follows.

(i)each element of G is a vertex of Cay(G,S)

(ii)for x and y in G there is an arc from x to y if and only if xs=y for some s ϵS.

Definition 1.3: A Cayley digraph of a group G with generating set S denoted by Cay(G,S) is said to be normal if R(G) is normal in Aut(Cay(G,S)). Where R(G) is the right regular representation of G and Aut(Cay(G,S)) is the automorphism group of graph Cay(G,S).

Remark:1.4: The right regular representations of a group is a subgroup of the group of Automorphisms of the group

Remarks.1.5: Cayley graphs are those graphs whose vertices may be identified with elements of groups and adjacency relation may be defined by subsets of the group. There are several ways to draw the digraph of a group given by a particular generating set. Headless arrows joining two vertices x and y indicates that there is an arc from x to y and an arc from y to x. This occurs when the generating set contains both an element and its inverse.

Definition:1.6: A quotient graph Q of a graph G is a graph whose vertices are blocks of a partition of the vertices of G and where block B is adjacent to block C if some vertex in B is adjacent to some vertex in C with respect to the edge set of G. In other words, if G has edge set E and vertex set V and R is the equivalence relation induced by the partition, then the quotient graph has vertex set V/R and edge set $\{([u]_R, [v]_R) \mid (u, v) \in E(G)\}$.

Definition:1.7: A cayley graph Cay(G,S) is said to be normal if R(G) is normal in Aut(Cay(G,S)).

Remarks.1.8. The Aut(Cay(G,S)) is the automorphism group of graph Cay(G,S) is a subset of Aut(G) satisfying the adjacency constraints on the set of vertices of the graph Cay(G,S) .

Remarks.1.9. It is well known that the cayley graph of a prime order is normal if the graph is neither the empty graph nor the complete graph.

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2. The normality of quotient graph in the Cayley graph of Solvable Group.

In paper [5] the author proved that ,the Cayley graph X= Cay(G,S) of the solvable group of derived length n contains the quotient graph $\bar{X}$ of the corresponding factor group G/H_{n-1} where S is a generating set of G.

The following theorem proves that the Quotient graph in the Cayley graph of a solvable group is normal.

Theorem:2.1: The quotient graph $\bar{X}$ of the corresponding factor group G/H_{n-1} in the Cayley graph X= Cay(G,S) of the solvable group is normal.

Proof: Let G be a solvable group and let S be a generating set of G such that e does not belong to S. Let H_{n-1}= N, the normal subgroup N which is normal subgroup of G in composition series $\{H_i\}$ such that G/N is simple.

To prove the theorem we have to prove that R(G/N), the right regular multiplications of G/N is normal in Aut(Cay(G/N, $\bar{S}$)).

Let $\sigma \epsilon$ Aut(Cay(G/N, , $\bar{S}$)).then

$\sigma^{-1} \epsilon$ Aut(Cay(G/N, $\bar{S}$))). Since Aut(Cay(G/N, , $\bar{S}$))) is a subset of Aut(G/N) which is a group.

Let $\rho \epsilon$ R(G/N) wehave to prove that $\sigma \rho \sigma^{-1} \epsilon$ R(G/N)

For that $xH(\sigma\ \rho\ \sigma^{-1}) = xH(\sigma)\quad xH(\rho)\quad xH(\sigma^{-1})$, since Aut(G/N) is a group under composition of mappings.

Also let $xH(\sigma) = yH$ for some $y \in G$,Then

$xH(\sigma^{-1}) = (xH(\sigma))^{-1} = (xH)^{-1}\sigma = x^{-1}H(\sigma) = y^{-1}H$. Since σ is an automorphism of G/H.

And let $xH(\rho) = xH\ aH$ for $\rho \in R(G/N)$, Then

$$xH(\sigma\rho\sigma^{-1}) = xH(\sigma)\quad xH(\rho)\quad xH(\sigma^{-1})$$

$$= yH\ xHaH\ y^{-1}H$$

$$= y(xa)\ y^{-1}H \in G/N$$

$= xay\ y^{-1}H$, since G/N is commutative ,since G is solvable.

$$= xaH$$

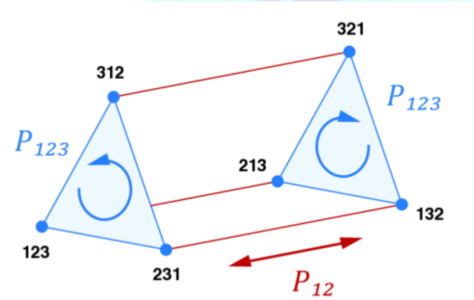
$$= xH\ aH$$

$$= xH(\rho)\ ,\quad \rho \in R(G/N)$$

R(G/N), the right regular multiplications of G/N is normal in Aut(Cay(G/N, $\bar{S}$)).

Hence the theorem

Example.2.2.Given below is an example of the Cayley graph of the group S_3 and $S_{3/}A_3$ .In this example $R(S_{3/}A_3)$ the same as Aut(Cay($S_{3/}A_3$)).

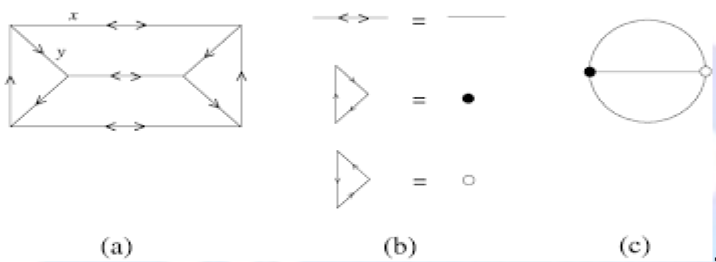


Cayley graph of $S_3$with generating set {(123),(132),(12)} .

:Fig(1)

The red edges between the vertices indicate they are double edges. The blue edges are single edges with the direction given in the figure.

$G = S_3$, and then $A_3=\{e,(123),(132)\}$.
Then $S_3/A_3=\{A_3,(12)A_3\} = \{\{e,(123),(132)\},\{(12),(23),(13)\}\}$.
$Cay(S_3/A_3, \{A_3,(12)A_3\})$ is given in Fig (3) with explanation.



Fig(2)

In Fig(2) :(a) is Cayley graph if S_3 , in (b) the line shows the edges are double edges , the dark vertex represents the block $\{e,(123),(132)\}$ and the bubble represents the block $\{(12),(23),(13)\}$. (c)is the Cayley graph of S_3/A_3.

The Figures in this paper are from the site [9]

3.Conclusion

In this paper the characteristic of quotient graph in the cayley graph of a solvable group .

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