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IRREDUCIBILITY OF POLYNOMIALS WITH INTEGRAL CO-EFFICIENTS

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Abstract: In this paper I have proved some theorems based on irreducibility of polynomials. I have taken different cases for functions to be irreducible over different fields such as rationals, integers etc. Different assumptions are chosen for the coefficients of polynomials Theorem of A. Cohn on irreducibility is main result.

Lemma 1.1 : Let k and b be integers with $1 \leq k \leq b$ and let

$$D = \left[z : |b - z| \leq \sqrt{k}\right]$$

Suppose $f(x) = \sum_{j=0}^{n} a_j x^j \in z[x]$

has the properties,

(i) $a_j \geq 0$ for j = 0, 1, ...n

(ii) $|f(b)| = kp$ for some prime p

(iii) if f(α) = 0 then $\alpha \notin D$

Then f(x) is irreducible over the rationals.

Proof: Assume f(x) satisfies (i), (ii), (iii). Then it is enough to prove that f(x) is irreducible over the integers. If possible, let f(x) be reducible i.e.

let f(x) = g(x) h(x)

with degree $(g(x) = r \geq 1)$

and degree $(h(x) = s \geq 1)$.

We also assume that leading coefficients of g(x) and h(x) are positive. For irreducibility we have to show that one of g(x) or h(x) is of degree zero. Now since

$$kp = |f(b)| = |g(b)\,h(b)| = |g(b)|\,|h(b)|$$

$\Rightarrow$ either p/g(b) or p/h(b).

Let us assume

$$p / |h(b)|$$

So $\dfrac{k}{|g(b)|} = \dfrac{|h(b)|}{p}$

$\Rightarrow$ $|g(b)| / k$

$\Rightarrow$ $|g(b)| \le k$

write g(x) = $b_r . \prod_{j=1}^{r} (x - \beta_j)$

Since each root β_j of g(x) is a root of f(x) i.e. $f(\beta_j)$ = 0, so by (iii) property

$\beta_j \notin D$

$\Rightarrow$ $|b - \beta_j| > \sqrt{k}$ for j = 1, 2, 3, ... r.

Hence k ≥ $|g(b)| = |b_r| \prod_{j=1}^{r} |(b - \beta_j)|$

$> k^{r/2}$

$\Rightarrow$ $k > k^{r/2}$

so that r = 1. Thus

$g(x) = b_1 (x - \beta_1)$

$= b_1 x - b_1\beta_1$ with $b_1 \ge 1$ since leading coefficient of g(x) is positive, b_1 is integer. Now if $\beta_1 > 0$ then g(x) has one real root which is positive.

Thus g(x) and therefore f(x) has a positive real root but this is a contradiction since we have given

$a_j \ge 0$ for j = 0, 1, ..., n.

$\Rightarrow$ f(x) has no positive real roots.

$\Rightarrow$ So $\beta_1 < 0$ hence $-b_1\beta_1 \ge 0$,

$b > k \ge |g(b)| = |b_1 b - b_1\beta_1| \ge b_1 b$

$\Rightarrow$ $b > b_1 b$ gives a contradiction. So our supposition is wrong.

Hence f(x) is irreducible over the rationals.

Remark 1.1 : A result similar to this lemma (where the coefficient of f(x) are allowed to be negative) can be found.

Theorem 1.1 : Let f(x) = $\sum_{j=0}^{n} a_j x^j \in Z[x]$

such that $a_j \geq 0$ for j = 0, 1, 2, ..., n

and let k and b be positive integers with b ≥ 2 and k ≤ b – 1.

Let $m = \left[\dfrac{\pi}{\sin^{-1} \dfrac{\sqrt{k}}{b}} \right] - 1$

where [ ] denotes the greatest integer function and fix $B \leq \gamma \left(b - \sqrt{k}\right)^m \left(b - \sqrt{k} - 1\right)$

where $\gamma = \dfrac{a_n \sqrt{k}}{\sqrt{b^2 - k} + \sqrt{k}}$.

Suppose that $a_j \leq B$ for j = 0, 1, ... n – m – 1

and f(b) = wp for some positive integer w ≤ k and some prime p. Then f(x) is irreducible over rationals.

Proof : Define a set by

$$S = \left\{ \alpha = re^{i\theta} : r \geq b - \sqrt{k} \text{ and } |\theta| \leq \sin^{-1}\left(\frac{\sqrt{k}}{b}\right) \right\}.$$

Now we will show that if $\alpha \in S$ then f(α) ≠ 0 and then we will apply lemma 1.1. Set

$$D = \left[z : |b - z| \leq \sqrt{k} \right]$$

Let z ∈ D. Now

$$\begin{aligned} b - |z| &= |b| - |z| \\ &\leq |b - z| \\ &\leq \sqrt{k} \end{aligned}$$

$\Rightarrow \quad |z| \geq b - \sqrt{k}$

$\Rightarrow \quad |re^{i\theta}| \geq b - \sqrt{k}$

$\Rightarrow \quad |r||e^{i\theta}| \geq b - \sqrt{k}$

$\Rightarrow \quad r \geq b - \sqrt{k}$

$\Rightarrow \quad z \in S.$

Thus S contains D. Now let $\alpha = re^{i\theta} \in S$ and $f(\alpha) = 0$.

i.e. $a_0 + a_1\alpha + \dots\dots\dots + a_n\alpha^n = 0.$

$a_0 + a_1 re^{i\theta} + \dots\dots\dots\dots + a_n r^n e^{in\theta} = 0.$

But $a_0, a_1, \dots\dots\dots\dots, a_n \geq 0$ and $r > 0$.

So $\theta \neq 0$.

Thus α is not real and since complex roots occur in pairs so conjugate of α is also an element of S.

We may therefore assume $\theta > 0$.

Set $m' = \left[\frac{\pi}{2\theta}\right]$ where [ ] is greatest integer function. Define $\theta_0 = \sin^{-1}\left(\frac{\sqrt{k}}{b}\right)$. Then

$$m = \left[\frac{\pi}{\theta_0}\right] - 1.$$

Set $\theta_1 = \pi - m\theta_0$.

Now, we will consider two cases :-

(i) $m \geq m'$ (ii) $m < m'$

consider the first case $m \geq m'$. Then

$$j\theta \in \left(0, \frac{\pi}{2}\right) \quad \text{for } j = 1, 2, \dots\dots\dots, m'$$

and $$j\theta \in \left(\frac{\pi}{2}, \pi\right) \quad \text{for } j = m' + 1, \dots\dots\dots, m.$$

Indeed $m\theta \leq \left[\frac{\pi}{\theta_0} - 1\right]\theta \leq \pi - \theta_0$ since $\theta \leq \theta_0$.

Using $\alpha^{-j} = r^{-j} e^{i(-j\theta)}$

$$= r^{-j}\cos(-j\theta) + i\sin(-j\theta).$$

We get

(1) $\operatorname{Re}(\alpha^{-j}) \geq 0$ for $j = 1, 2, \dots\dots\dots, m'$

(2) $\operatorname{Re}(\alpha^{-j}) < 0$ for j = m′ + 1, ..., m

since $r^{-j}\sin(-j\theta) = -r^{-j}\sin j\theta$ for all values of θ and $\operatorname{Im}(\alpha^{-j}) < 0$ for j = 1, ..., m. --(3)

Now, $\theta_1 \geq \theta_0$ because

$$\pi - m\theta_0 \geq \theta_0$$

$$\Leftrightarrow \quad \pi \geq (m+1)\theta_0$$

$$\Leftrightarrow \quad \frac{\pi}{m+1} \geq \theta_0$$

$$\Leftrightarrow \quad \theta_0 \leq \frac{\pi}{m+1}$$

and $\theta_1 = \pi - m\theta_0 \geq \pi - \frac{\pi}{m+1} m \geq \frac{\pi}{m+1}$

So $\theta_0 \leq \frac{\pi}{m+1}$ and $\theta_1 \geq \frac{\pi}{m+1}$

$$\Rightarrow \quad \theta_1 \geq \theta_0$$

since $\theta \leq \theta_0$

$$-j\theta \geq -m\theta_0 > 0 \quad \text{for} \quad j = m' + 1, \ldots\ldots\ldots, m$$

$$\frac{\pi}{2} > \pi - j\theta \geq \pi - m\theta_0 > 0$$

so $\tan(\pi - j\theta) \geq \tan(\pi - m\theta_0) = \tan\theta_1$

and since $\frac{\pi}{2} > \theta_1 \geq \theta_0$

$$\Rightarrow \quad \tan\theta_1 \geq \tan\theta_0$$

Thus $\tan(\pi - j\theta) \geq \tan\theta_1 \geq \tan\theta_0$ for j = m′ + 1, ..., m.

Now $|\operatorname{Im}(\alpha^{-j})| = |r^{-j}\sin(-j\theta)|$

$$= |r^{-j}\sin j\theta|$$

$$= r^{-j}\sin j\theta \quad (\because j\theta \in (0, \pi))$$

$$= r^{-j}\sin(\pi - j\theta)$$

$$= r^{-j}\tan(\pi - j\theta)\cos(\pi - j\theta)$$

$$\geq r^{-j}\tan\theta_0\cos(\pi - j\theta)$$

$$= r^{-j}\cos(\pi - j\theta)\tan\theta_0$$

$$= \left|\mathrm{Re}(\alpha^{-j})\right|\frac{\sqrt{k}}{\sqrt{b^2 - k}} \qquad \text{for} \qquad j = m' + 1, \ldots\ldots, -m$$

$$\Rightarrow \qquad |\mathrm{Im}(\alpha^{-j})| \geq \left|\mathrm{Re}(\alpha^{-j})\right|\frac{\sqrt{k}}{\sqrt{b^2 - k}} \tag{4}$$

We consider

$$\left|\frac{f(\alpha)}{\alpha^n}\right| = \left|a_n + a_{n-1}\alpha^{-1} + a_{n-1}\alpha^{-2} + \ldots\ldots + a_{n-m}\alpha^{-m} + \sum_{j=m+1}^{n} a_{n-j}\alpha^{-j}\right| \tag{5}$$

where (in the case that n < m) we interpret a_{n-j} as zero for j > n and the sum is zero.

Now if $\left|\mathrm{Re}\left(a_{n-m'-1} - \alpha^{-m'-1} + \ldots\ldots + a_{n-m}\alpha^{-m}\right)\right| \leq a_n - \gamma$

Then we have by (1) and (5).

$$\left|\frac{f(\alpha)}{\alpha^n}\right| \geq \left|a_n + a_{n-1}\alpha^{-1} + \ldots\ldots + a_{n-m}\alpha^{-m}\right| - \sum_{j=m+1}^{n} |a_{n-j}||\alpha|^{-j}$$

$$\geq \mathrm{Re}\left|a_n + a_{n-1}\alpha^{-1} + \ldots\ldots + a_{n-m}\alpha^{-m}\right| - \sum_{j=m+1}^{n} Br^{-j}$$

$$> \mathrm{Re}\left|a_n + a_{n-1}\alpha^{-1} + \ldots\ldots + a_{n-m'}\ \alpha^{-m'}\right|$$

$$-\left|\mathrm{Re}\left(a_{n-m'-1}\alpha^{-m'-1} + \ldots\ldots + a_{n-m}\alpha^{-m}\right)\right| - \sum_{j=m+1}^{n} Br^{-j}$$

$$\geq a_n - (a_n - \gamma) - B(r^{-m-1} + r^{-m-2} + \ldots\ldots\ldots\infty)$$

Since n < m or m > n ≥ m´

$\Rightarrow$ n ≥ m´

so $a_{n-m'}$ are all zero

$$\geq \gamma - \frac{B}{r^{m(r-1)}} \tag{6}$$

Now $\sqrt{\frac{k}{b^2-k}}(a_n-\gamma)$

$$= \sqrt{\frac{k}{b^2-k}}\left(a_n - \frac{a_n\sqrt{k}}{\sqrt{b^2-k}+\sqrt{k}}\right)$$

$$= \frac{a_n\sqrt{k}}{\sqrt{b^2-k}}\left\{\frac{\sqrt{b^2-k}+\sqrt{k}-\sqrt{k}}{\sqrt{b^2-k}+\sqrt{k}}\right\}$$

$$= \frac{a_n\sqrt{k}}{\sqrt{b^2-k}+\sqrt{k}} = \gamma$$

$$\therefore \quad \left\{\frac{k}{b^2-k}\right\}^{1/2}(a_n-\gamma)=\gamma$$

Now if $\mathrm{Re}\left(a_{n-m'-1}-\alpha^{-m'-1}+\dots\dots..+a_{n-m}\alpha^{-m}\right) > a_n-\gamma$

Then $\left|\frac{f(\alpha)}{\alpha^n}\right| \geq \left|a_n+a_{n-1}\alpha^{-1}+\dots\dots\dots+a_{n-m}\alpha^{-m}\right| - \sum_{j=m+1}^{n}|a_{n-j}||\alpha|^{-j}$

$$> \left|a_n+a_{n-1}\alpha^{-1}+\dots\dots\dots+a_{n-m}\alpha^{-m}\right| - \sum_{j=m+1}^{\infty}Br^{-j}$$

$$\geq \left|\mathrm{Im}\left(a_{n-m'-1}\alpha^{-m'-1}+\dots\dots..+a_{n-m}\alpha^{-m}\right)\right| - \sum_{j=m+1}^{\infty}Br^{-j} \quad \text{(by (2) \& (3))}$$

$$> \frac{\sqrt{k}}{\sqrt{b^2-k}}(a_n-\gamma) - \sum_{j=m+1}^{\infty}Br^{-j} \quad \text{(by (4))}$$

$$= \gamma - \frac{B}{r^m(r-1)} \quad (7)$$

Thus in any case $\left|\frac{f(\alpha)}{\alpha^n}\right| > \gamma - \frac{B}{r^m(r-1)}$.

Since $r \geq b-\sqrt{k}$ and $B \leq \gamma\left(b-\sqrt{k}\right)^m\left(b-\sqrt{k}-1\right)$

$$\left|\frac{f(\alpha)}{\alpha^n}\right| > \gamma - \frac{B}{r^m(r-1)}$$

$$\geq \gamma - \frac{\gamma\left(b-\sqrt{k}\right)^m\left(b-\sqrt{k}-1\right)}{r^m(r-1)}$$

$$= \gamma\left[1 - \frac{\left(b-\sqrt{k}\right)^m\left(b-\sqrt{k}-1\right)}{r^m(r-1)}\right]$$

$$= \gamma\left[\frac{\left(b-\sqrt{k}\right)^{m+1} - \left(b-\sqrt{k}\right)^m - \left(b-\sqrt{k}\right)^{m+1} + \left(b-\sqrt{k}\right)^m}{r^m(r-1)}\right]$$

$$= 0$$

$\Rightarrow$ $|f(\alpha)| > 0$ which is a contradiction because we have assumed $f(\alpha) = 0$.

In the case $m < m'$, (1) holds for $j \leq m$

i.e. $\mathrm{Re}\left(\alpha^{-j}\right) \geq 0$ for $j = 1, 2, \ldots\ldots\ldots, m$.

$$\left|\frac{f(\alpha)}{\alpha^n}\right| > a_n - \frac{B}{r^m(r-1)}$$

$$\geq \gamma - \frac{B}{r^m(r-1)} > 0$$

so $|f(\alpha)| > 0$ which is a contradiction. Thus if $\alpha \in S$ then $f(\alpha) \neq 0$. Since all the conditions of Lemma 1.1 hold so by lemma 1, we get f(x) is irreducible over rationals.

Comments

1.1.1 Theorem can be used to test the irreducibility of an arbitrary polynomial if b is choosen sufficiently large and a translation is made so that the coefficient of the polynomial are non-negative.

Since if α_j's are roots of f(x) and if

$|\mathrm{Re}(\beta_j)| < B$ for all j,

then we know that f(x + B) has non-negative coefficients and then by taking b sufficiently large, we can apply Theorem 1.1.

1.1.2 In theorem 1.1, we can choose γ as

$$\gamma = a_n\left|\left(1+\cot\left(\theta_1\right)\right)\right|$$

since from theorem 1.1, we have

$$\gamma = \frac{a_n\sqrt{k}}{(b^2-k)^{\frac{1}{2}}+\sqrt{k}}$$

Set $\sqrt{k} = b\sin\theta_1$

Then $\gamma = \dfrac{a_n b\sin\theta_1}{b\cos\theta_1 + b\sin\theta_1}$

$= \dfrac{a_n}{\cot\theta_1 + 1}$ (By dividing by b $\sin\theta_1$ in denominator and numerator)

Theorem 1.2 : Let f(x) = $\sum_{j=0}^{n} a_j x^j \in Z[x]$ be such that f(10) is a prime. If $0 \le a_j \le a_n 10^{30}$ for each j = 0, 1, ..., n-1 then f(x) is irreducible.

Proof : To prove this theorem, we use ideas from the proof of theorem 1.1. Take b = 10 and k = 1.

Write T = S_1 U S_2 U S_3 where

$$S_1 = \left\{\alpha = re^{i\theta} : r \ge 9.75, \frac{\pi}{32} < |\theta| \le \sin^{-1}\left(\frac{1}{10}\right)\right\}$$

$$S_2 = \left\{\alpha = re^{i\theta} : r \ge 9.64, \frac{\pi}{33} < |\theta| \le \frac{\pi}{32}\right\}$$

$$S_3 = \left\{\alpha = re^{i\theta} : r \ge 9, 0 < |\theta| \le \frac{\pi}{33}\right\}$$

Then D = $\{Z : |10 - z| < 1\} \subset T$

We need only to show for all $\alpha \in$ T, f(α) $\neq$ 0. Assume for some $\alpha = re^{i\theta} \in$ T, f(α) = 0. Since the coefficients of f(x) are non-negative so $\theta \neq 0$. We may take $\theta > 0$. From the proof of Theorem 1.1, we can arrive at by (6) and (7), $\left|\dfrac{f(\alpha)}{\alpha^n}\right| > \gamma - \left\{\dfrac{B}{r^m(r-1)}\right\}$.

Here choice of m and γ depends to which set S_j, j = 1, 2, or 3 α belongs with B ≤ $a_n 10^{30}$.

For S_1, we may take $\gamma = a_n$ (0.035)

and m = 31

For S_2, we may take $\gamma = a_n$ (0.089)

and m = 31

For S_3, we may take $\gamma = a_n(0.087)$

and $m = 32$

$$\left|\frac{f(\alpha)}{\alpha^n}\right| > \gamma - \frac{B}{r^n(r-1)}$$

For S_1, $\left|\frac{f(\alpha)}{\alpha^n}\right| \geq a_n(0.035) - \frac{a_n 10^{30}}{r^{31}(r-1)}$

$$= a_n\left[0.035 - \frac{10^{30}}{(9.75)^{31}(9.75-1)}\right]$$

$$= a_n\left[0.035 - \frac{1}{39.916379}\right]$$

> 0

For S_2, $\left|\frac{f(\alpha)}{\alpha^n}\right| \geq a_n\left[0.089 - \frac{10^{30}}{(9.64)^{31}(9.64-1)}\right]$

$$= a_n\left[0.089 - \frac{1}{27.72693}\right]$$

> 0

Similarly for S_3, we get $\left|\frac{f(\alpha)}{\alpha^n}\right| > 0$.

Thus in each case, we have

$|f(\alpha)| > 0$ with $B \leq a_n 10^{30}$

which gives a contradiction since $f(\alpha) = 0$.

Thus for all $\alpha \in T$, $f(\alpha) \neq 0$.

Hence all the conditions of theorem 2 are true so f(x) is irreducible over rationals.

Lemma 1.2 : If $f(x) \in Z[x]$ is of degree $n \geq 1$ and has non-negative coefficients then it has no zero in the sector $S = \left[Z = \rho e^{i\theta} : \rho > 0, |\theta| < \frac{\pi}{n}\right]$.

Proof : Let $Z = \rho e^{i\theta} \in S$. If Z is zeal then since f(x) has non-negative coefficients so $f(Z) \neq 0$. Suppose Z is not real then either $Im(Z^j) > 0$ for all j = 1, 2, ..., n or $Im(Z^j) < 0$ for all j = 1, 2, ...n.

Now $f(Z) = a_0 + a_1Z + a_2Z^2 + \ldots\ldots.. + a_nZ^n$

$= a_0 + a_1\rho e^{i\theta} + a_2\rho^2 e^{2i\theta} + \ldots\ldots\ldots.. + a_n\rho^n e^{in\theta}$

Im f(Z) = $a_1\rho\sin\theta + a_2\rho^2\sin 2\theta + \ldots\ldots.. + a_n\rho^n\sin n\theta$

$$|\operatorname{Im} f(Z)| = |a_1\rho\sin\theta + a_2\rho^2\sin 2\theta + \ldots\ldots.. + a_n\rho^n\sin n\theta|$$

If $\operatorname{Im}(Z^j) > 0$ that is $\rho^j \sin j\theta > 0$ for $j = 1, 2, \ldots\ldots\ldots., n$,

then $|\operatorname{Im} f(Z)| > 0$.

If $\operatorname{Im}(z^j) < 0$ that is $p^j \sin j\theta < 0$ for $j = 1, 2$ -, ..., n,

then $|\operatorname{Im} f(z)| = |-(a_1\rho \sin\theta + a_2 \rho^2 \sin 2\theta + \ldots\ldots\ldots\ldots. + a_n\rho \sin n\theta|$

$= a_1 \rho \sin \theta + a_2 \rho^2 \sin 2\theta + \ldots\ldots\ldots. + a_n \rho^n \sin n\theta$

> 0 (since $a_j \geq 0$ for $j = 1, 2, \ldots\ldots\ldots, n$)

Thus $|\operatorname{Im} (fZ)| > 0$.

So $f(z) \neq 0$ because for f(z) to be zero, it's real and imaginary part must be equal to zero.

Hence we have shown for $z \in S$,

$f(z) \neq 0$, i.e. f(x) has no zero in the sector.

$$S = \left\{z = \rho e^{i\theta} : \rho > 0, |\theta| < \frac{\pi}{n}\right\}.$$

Theorem 1.3 : Let $b > 1$ be an integer and let $N_1 = \dfrac{\pi}{\sin^{-1}\left(\frac{1}{b}\right)}$. If $f(x) \in Z[x]$ is of degree $n < N_1$ and has non-negative coefficients and if f(b) is prime then f(x) is irreducible.

Proof : We prove this theorem by using lemma 1.1. In order to prove this we have to show that f(x) has no zeros in the set

$S = \{z : |b - z| \leq 1\}$

Now $\{z : |b - z| \leq 1\} \subset \left\{z = \rho e^{i\theta} : \rho > 0, |\theta| \leq \sin^{-1}\left(\frac{1}{b}\right)\right\} \subset \left\{z = \rho e^{i\theta} : \rho > 0, |\theta| < \frac{\pi}{n}\right\}$.

But by lemma 1.2, we have proved that f(x) has no zeros in the sect

$$\left\{z = \rho e^{i\theta} : \rho > 0, |\theta| < \frac{\pi}{n}\right\}.$$

So f(x) has no zeros in the set S.

Then by lemma 1.1, f(x) is irreducible.

Lemma 1.3 : If g(x) $\in$ z [x] has no non-negative real roots then there is an h(x) $\in$ z[x] such that g(x) h(x) has all positive coefficients.

Proof : Suppose g(x) $\in$ z[x] has no non-negative real roots. Then the constant term of g(x) is non zero. Since g(x) h(x) is to have positive coefficients, we consider only

h(x) $\in$ z[x] with non-zero constant terms.

Now if g(x) h(x) = $\sum_{j=0}^{n} a_j x^j \in Z[x]$

has non-negative coefficients then for a suitable positive integer d, we get

$$g(x)h(x)\left(x^d + x^{d-1} + \dots\dots\dots + 1\right)$$

has all positive coefficients. Thus we have to show that for given g(x) $\in$ Z[x] there is an h(x) $\in$ Z[x] with real coefficients such that g(x)h(x) has non-negative coefficients.

Write

$$g(x) = \sum_{j=0}^{r} b_j x^j$$

and let R be a positive integer such that Rg(x) h(x) has all its coefficients greater than

$$\sum_{j=0}^{r} |b_j|$$

write

$$Rh(x) = \sum_{j=0}^{s} \gamma_j x^j$$

and set $C_j = [\gamma_j]$ where [ ] denotes the greatest integer function. Let

$$H(x) = \sum_{j=0}^{s} c_j x^j \in Z[x].$$

Then g(x) H(x) = g(x) $\left\{ Rh(x) - \sum_{j=0}^{s} (\gamma_j - c_j) x^j \right\}$ (1)

Since $\quad Rh(x) - \sum_{j=0}^{s} (\gamma_j - c_j) x^j$

$$= \sum_{j=0}^{s} \gamma_j x^j - \sum_{j=0}^{s} (\gamma_j - c_j) x^j$$

$$= (\gamma_0 + \gamma_1 x + \ldots\ldots + \gamma_s x^s) - [(\gamma_0 - c_0) + (\gamma_1 - c_1)x + \ldots\ldots + (\gamma_s - c_s)x^s]$$

$$= c_0 + c_1 x + c_2 x^2 + \ldots\ldots\ldots\ldots + c_2 x^s$$

$$= \sum_{j=0}^{s} c_j x^j$$

$$= H(x)$$

$$\Rightarrow \quad Rh(x) - \sum_{j=0}^{s} (\gamma_j - c_j) x^j = H(x).$$

From (1), we get

$$g(x)\, H(x) = Rg(x)\, h(x) - g(x) \left\{ \sum_{j=0}^{s} (\gamma_j - c_j) x^j \right\}$$

which has all positive coefficients since each coefficient of

$$g(x) \left\{ \sum_{j=0}^{s} (\gamma_j - c_j) x^j \right\} \leq \sum_{j=0}^{r} | b_j | \max_{0 \leq j \leq s} | \gamma_j - c_j |$$

and these b_j are positive.

Write

$$g(x) = b_r \prod_{j=1}^{u} (x + \beta_j) \prod_{j=1}^{v} \left(x^2 - 2r_j (\cos\theta_j)\, x + r_j^2\right)$$

where b_r is an integer, $\beta_j > 0$ for j = 1, 2, 3, ..., u and $r_j > 0$ and $0 < \theta_j < \pi$ for j = 1, 2, ..., v. Let $k = \rho e^{i\theta}$ and $\lambda = \rho e^{-i\theta}$ where $\rho > 0$ and $0 < \theta < \Pi$.

We need only show that for such k and λ there is an $h(x) \in R[x]$ (depending on k and λ) such that

$$(x - k)\,(x - \lambda)\, h(x)$$

has all non-negative coefficients. Let s be the non-negative integer satisfying

$\left(\frac{\pi}{\theta}\right)-2 \le s \le \left(\frac{\pi}{\theta}\right)-1$ for j = 0, 1, 2 ...., s, define

$$c_j = \left(k^{s-j+1} - \lambda^{s-j+1}\right) / (k-\lambda) \tag{2}$$

$$c_j = \frac{\rho^{s-j+1} e^{i(s-j+1)\theta} - \rho^{s-j+1} e^{-i(s-j+1)\theta}}{\rho e^{i\theta} - e^{-i\theta}}$$

$$= \frac{\rho^{s-j+1}[\cos(s-j+1)\theta + i\sin(s-j+1)\theta] - \rho^{s-j+1}[\cos(s-j+1)\theta - i\sin(s-j+1)\theta]}{\rho(\cos\theta + i\sin\theta) - \rho(\cos\theta - i\sin\theta)}$$

$$= \frac{\rho^{s-j+1}[\cos(s-j+1)\theta + i\sin(s-j+1)\theta - \cos(s-j+1)\theta + i\sin(s-j+1)\theta]}{\rho(\cos\theta + i\sin\theta - \cos\theta - i\sin\theta)}$$

$$= \frac{\rho^{s-j+1}[2\, i\sin(s-j+1)\theta]}{\rho(2\, i\sin\theta)}$$

$$= \frac{\rho^{s-j}\sin(s-j+1)\theta}{\sin\theta}$$

> 0 for j = 1, 2, ..., s.

Let h(x) = $\sum_{j=0}^{s} c_j x^j$.

Then (x – k) (x - λ) h(x) = [x^2 – (k + λ) x + kλ] (h(x))

= [x^2 – (k + λ) x + kλ] (c_0 + c_1x, ..., $c_s x^s$)

But from (2), $c_j = \frac{k^{s-j+1} - \lambda^{s-j+1}}{k-\lambda}$

$$c_0 = \frac{k^{s-j+1} - \lambda^{s+1}}{k-\lambda}$$

$$c_1 = \frac{k^s - \lambda^s}{k-\lambda}$$

$$c_s = \frac{k-\lambda}{k-\lambda} = 1$$

So $[(x-k)(x-\lambda)]\, h(x) = (x^2-(k+\lambda)x+k\lambda)$

$$\left(\frac{\lambda k^{s+1}-\lambda^{s+1}}{k-\lambda}+\frac{k^s-\lambda^s}{k-\lambda}x+........+1.x\right)$$

$$= x^{s+2} - \frac{k^{s+2}-k\lambda^{s+1}+\lambda k^{s+1}-\lambda^{s+2}}{k-\lambda}x+\frac{k^{s+1}\lambda-k^{s+1}\lambda}{k-\lambda}x+\frac{k^{s+1}-\lambda^{s+1}}{k-\lambda}.k\lambda$$

$$= x^{s+2} - \frac{\left(k^{s+2}-\lambda^{s+2}\right)\times}{k-\lambda}+\frac{\left(k^{s+1}-\lambda^{s+1}\right)k\lambda}{k-\lambda}$$

But $$\frac{k^{s+2}-\lambda^{s+2}}{k-\lambda}=\frac{\rho^{s+2}e^{i(s+2)}\phi-\rho^{s+2}e^{i(s+2)}\theta}{\rho e^{i\theta}-\rho e^{-i\theta}}$$

$$= \frac{\rho^{s+2}\left[\cos(s+2)\theta+i\sin(s+2)\theta-\cos(s+2)\theta+i\sin(s+2)\theta\right]}{\rho[\cos\theta+i\ \sin\theta-\cos\theta+i\ \sin\theta]}$$

$$= \frac{\rho^{s+2}\left[2i\sin(s+2)\theta\right]}{\rho[2i\ \sin\theta]}$$

$$= \frac{\rho^{s+1}\sin(s+2)\theta}{\sin\theta}$$

and $$\frac{k^{s+1}-\lambda^{s+1}}{k-\lambda}=\frac{\rho^{s+q}e^{i(s+q)}\phi-\rho^{s+q}e^{-i(s+q)}\theta}{\rho e^{i\theta}-\rho e^{-i\theta}}$$

$$= \frac{\rho^{s+1}\left[\cos(s+1)\theta+i\sin(s+1)\theta-\cos(s+1)\theta+i\sin(s+1)\theta\right]}{(\cos\theta+i\ \sin\theta-\cos\theta+i\ \sin\theta)}$$

$$= \frac{\rho^{s}\left[2i\sin(s+1)\theta\right]}{(2i\ \sin\theta)}$$

Hence $(x-k)(x-\lambda)(h(x) = x^{s+2} - \frac{\rho^{s+1}\sin(s+2)\theta}{\sin\theta}x - \rho^{s+2} - \frac{\sin(s+1)\theta}{\sin\theta}$.

But from choice of s, we get

$(x-k)(x-\lambda)(h(x))$ has non-negative coefficient.

Hence we have shown that there is $h(x) \in Z[x]$ such that $g(x)\, h(x)$ has all positive coefficients.

Lemma 1.4 : Let $b > 1$ be an integer and $N_2 = \pi \big/ \tan^{-1}\left(\frac{1}{b}\right)$.

Let n be the positive integer such that $N_2 \leq n < N_2 + 1$. Then there is a polynomial of degree n with non-negative coefficients and positive which is divisible by $(x - b)^2 + 1$.

Proof : Let $\rho = (b^2 + 1)$ and $\theta = \tan^{-1}\left(\frac{1}{b}\right)$.

Now if $k = \rho e^{i\theta}$, $\lambda = \rho e^{-i\theta}$ with the coefficient is

$$(x - k)(x - \lambda) = (x - \rho e^{i\theta})(x - \rho e^{-i\theta})$$

$$= x^2 - \rho e^{-i\theta} x - \rho e^{i\theta} x + \rho^2$$

$$= x^2 - \rho x (e^{-i\theta} + e^{i\theta}) + p^2$$

$$= x^2 - \rho x (2 \cos \theta) + \rho^2$$

$$= x^2 - \sqrt{b^2 + 1}\,(2 \cos \theta)\, x + b^2 + 1$$

But $\tan\theta = \left(\frac{1}{b}\right) \Rightarrow \cos\theta = \frac{b}{b^2 + 1}$

$$\therefore \quad (x - k)(x - \lambda) = x^2 - 2bx + b^2 + 1$$

$$= (x - b)^2 + 1$$

Taking $h(x) = \sum_{j=0}^{r} c_j x^j$ gives a polynomial form lemma 1.3

$$(x - k)(x - \lambda)\, h(x) = f(x)$$

$$\Rightarrow \quad ((x-b)^2 + 1)\, h(x) = f(x)$$

$$\Rightarrow \quad f(x) = (x - b)^2 + 1)\, h(x)$$

with non-negative coefficient of degree n. Now f(x) has already integral coefficients. Furthermore,

$$f(x) = (x - k)(x - \lambda)\, h(x)$$

$$= x^{s+2} - \rho^{s+1} - \frac{\sin(s+2)\theta}{\sin\theta} + \frac{\rho^{s+2}\sin(s+1)\theta}{\sin\theta} \qquad \text{(from lemma 1.3)}$$

Here f(x) is of degree $n = s + 2$. So the coefficient of x in f(x) is

$$-\frac{\rho^{n-1}\sin n\theta}{\sin\theta}$$

For $b \geq 2$, N_2 is not an integer.

Lemma 1.3 describes how to modify f(x) so as to obtain a polynomial of degree n which has non-negative integer coefficients and which is divisible by $(x – b)^2 + 1$.

Now $\pi = N_2\theta < n\theta < (N_2 + 1)\theta$

$= N_2\theta + \theta$

$= \pi + \theta$

$< 2\pi$.

So that the coefficients of x in f(x) is non-zero. Since $n\theta$ lies between Π and 2Π and sin e is negative in third and fourth quadrant. Thus coefficient of x in f(x) i.e. $\left(-\rho^{n-1}\left(-\frac{\sin n\theta}{\sin\theta}\right)\right)$ is positive and is divisible by $(x – b)^2 + 1$.

Hence the result.

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