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MATHEMATICS TEACHINGUSING THETENNISGAME

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ABSTRACT.Using sports in the teaching of mathematical and statistical concepts, students' interest enhances in topics like calculus, matrix algebra, Markov chains, probability, discrete & continuous probability distributions, descriptive statistics, regression, time series, sampling, and game theory. This article explains how a tennis game help as an aid for teaching calculus, statistics with an exposure to the basics of game theory.
Key Words:Average Velocity, Coefficient of variation,Deuce, expected pay-off, Markov chain, Mixed-Strategy, Nash Equilibrium, Probability of winning a game,

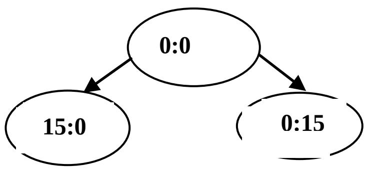
INTRODUCTION

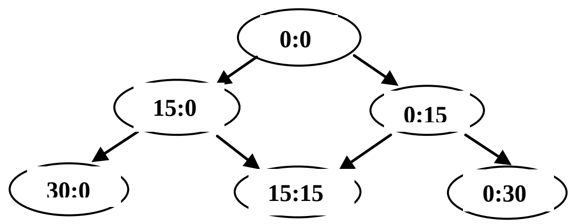
This article consists of three sections to illustrate how a single sport, *Tennis*, may be used toteach various concepts of *Statistics*, *Game theory*and *Calculus.* In the first section: *Statistical concepts of Tennis*, we introduce the concepts of probability, and conditional probability using the score points of the Tennis game. Some examples of comparing coefficients of variation, distributions and Markov chains will be discussed. In the second section: *Game Theory and Tennis*, we can use concepts of mixed-strategy Nash Equilibrium from game theory to predict the behavior of each player on the court, by determining their expected payoff. Game theory can also be applied to whether or not to challenge the line call during a tennis game. In the third section: *Calculus of Tennis*, we discuss how the tension (both main and crosses) of two rackets be compared using the average velocity of a tennis ball struck with rackets of varying tension.

1. STATISTICAL CONCEPTS OF TENNIS

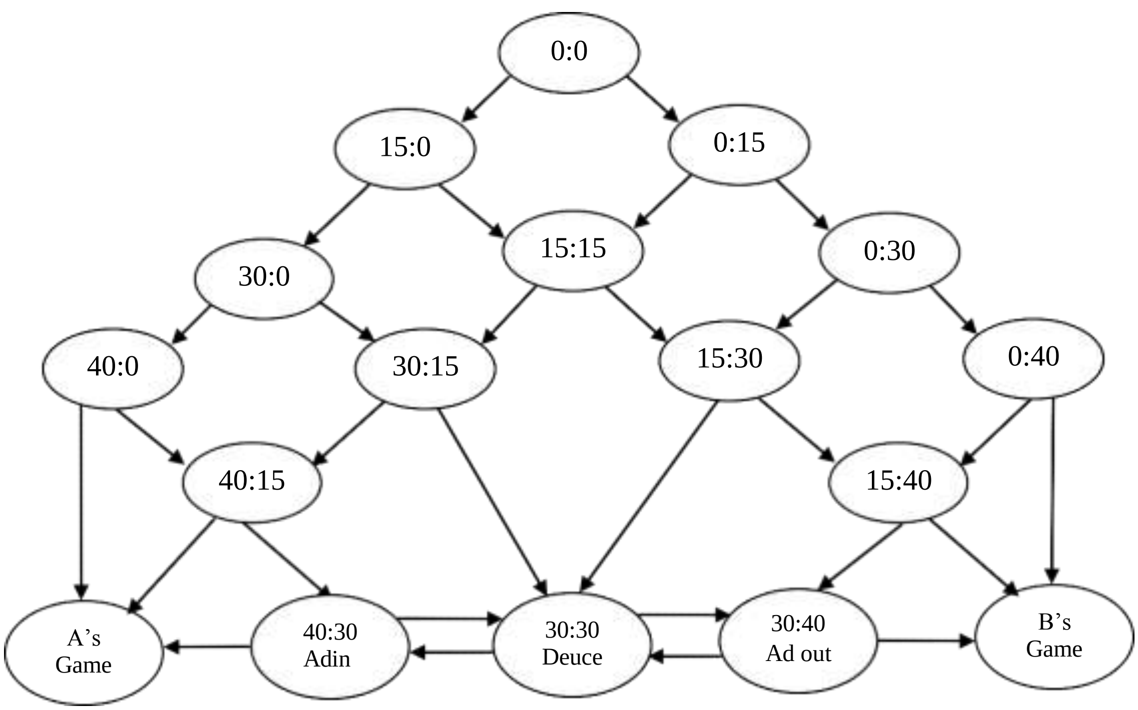
First we recallthe strange names used for points in scoring a game in tennis: *love*, *fifteen*, *thirty*, *forty*, *game*, and that a game must be won by lead of at least*two points*. (Note that fifteen, thirty, forty-five, sixty were originally used to represent the four quarters of an hour). If players each score *three points*, the score is called *deuce*, rather than 40:40. If the server wins the next point, the score becomes *advantage in* (Ad-in). If the server *wins* again, he/she *wins* the game, otherwise the score returns to *deuce*. If server *loses* the next point, the score becomes *advantage out* (Ad-out).If the server *loses* again, he/she *loses* the game, otherwise the score returns to *deuce*. This feature of tennis scoring increases the chance that the stronger player will win. Equivalently, we discuss the scoring pattern of the tennis game using graphical representation as below:

Consider a game of tennis between two players, A (the server) and B (the receiver). The progression of the game can be used to teach many statistical concepts and critical thinking. The score starts at (0:0). If the server wins the point at (0:0) then the score progresses to (15:0). If the receiver wins the point at (0,0) then the score progresses to (0:15). Therefore, the score progresses to either (15:0) or (0:15) from (0:0), shown below as the graphical representation of the *first point* played in a game.If the server wins the point at (15:0) then the score progresses to (30:0).If the receiver wins the point at (15:0) then the score progresses to (15:15). If the server wins the point at (0:15) then the score progresses to (15:15) and if the receiver wins the point at (0:15) then the score progresses to (0:30). Therefore, the score progresses to either (30:0) or (15:15) from (15:0), and the score progresses to either (15:15) or (0:30) from (0:15), shown below as the graphical representation of the *first two points* played in the game.





This process continues until either the server or receiver has won the game. To win a gamerequires winning *four points*. However, if the scores are level after *six points* have beenplayed (known as *deuce*) then play continues indefinitely until a player has established a two-point lead and wins the game. Hence the structure of a game can be broken up intotwo parts - winning a game prior to *deuce* and winning a game from *deuce*. For the serverto win the game prior to deuce requires winning 4 points whilst the receiver can win 0, 1or 2 points. We can draw figure to represent winning a game prior to deuce. A full game of tennis is represented in Figurebelow.



Suppose we assign a probability of the server winning a point. To keep the model simple, the probability is *constant* for every point of the game. Since the only two outcomes at eachscore within the game is either the server or receiver winning a point; the probability of the receiver winning a point is *one minus* the probability of the server winning a point.

For the game of tennis between two players, A(Server) and B(Receiver). For any event E, we use P(E) to denote the probability that E occurs.Letus define

$$p = \mathrm{P(Awinsapoint)}, q = 1 - p = \mathrm{P(Bwinsapoint)}$$

Question. *What is the probability of the server or receiver winning the game given the server has a probability p of winning a point? The first observation is since the game can only end with the server winning or the receiver winning, then the two probabilities combined must sum to* 1.

Answer. The server can only win to 0 (known as *love*) by obtaining *four* points in a row from (0:0). This occurs with probability p^4 as given by the following path:

$(0:0) \xrightarrow{p} (15:0) \xrightarrow{p} (30:0) \xrightarrow{p} (40:0) \xrightarrow{p}$ A (Server's game)

Theservercanwinto15fromthefollowing four possiblepaths, each with probability p^4q:

$(0:0) \xrightarrow{p} (15:0) \xrightarrow{p} (30:0) \xrightarrow{p} (40:0) \xrightarrow{q} (40:15) \xrightarrow{p}$ A (Server's game)
or
$(0:0) \xrightarrow{p} (15:0) \xrightarrow{p} (30:0) \xrightarrow{q} (30:15) \xrightarrow{p} (40:15) \xrightarrow{p}$ A (Server's game)
or
$(0:0) \xrightarrow{p} (15:0) \xrightarrow{q} (15:15) \xrightarrow{p} (30:15) \xrightarrow{p} (40:15) \xrightarrow{p}$ A (Server's game)
or
$(0:0) \xrightarrow{q} (0:15) \xrightarrow{p} (15:15) \xrightarrow{p} (30:15) \xrightarrow{p} (40:15) \xrightarrow{p}$ A (Server's game)

Since there are 4 paths, and the server and receiver win 4 points and 1 point respectively; the server wins the game to 15 with probability $p^4q + p^4q + p^4q + p^4q = 4p^4q$. Similarly,thereare10pathsfortheservertowin the game to30withprobability $10p^4q^2$.

Therefore,the probabilityofthe*server winningthegamewithout reachingdeuce*occurswithprobability$p^4+4p^4q+10p^4q^2=p^4(1+4q+10q^2)$.

Next, using the probability of the score *reaching deuce* occurs with probability 20p3q3 (since there are 20 paths that can occur in reaching deuce and each player wins exactly 3 points), we obtainthe following, adding the previous result and using $p + q = 1$, the probability of the server winning the game from the outset (0,0) is given by:

P(A wins the game) = $p^4(1+4q+10q^2) + 20p^3q^3[p^2/(p^2+q^2)] = p^4(1+4q+[10q^2/(p^2+q^2)])$.

Given $p = 0.6$, the probabilities of the server and receiver winning a game are calculated below, it shows that the server and receiver have a 0.736 and 0.264 of winning the game respectively.

Win to	Server	Receiver
0	0.130	0.026
15	0.207	0.061
30	0.207	0.092
Deuce	0.191	0.085
Game	0.736	0.264

Note: The Binomial distribution can also be applied to a tennis game since there are two outcomes; either the server wins a point or receiver wins a point, the probability of success on a single trial p of the server winning a point is constant from trial to trial, the probability of a failure is equal to $q = 1 - p$, and the game consists of n identical and independent trials.

Markov chains (a mathematical system that undergoes transitions from one state to another) have many applications as statistical models of real-world processes and can be applied to tennis as:

Thinkabout a game thathasreached thestatedeuce(or 30: 30).*There is no limit to howlongthegame couldgo on.From this point, the game couldreach oneofthe five possiblestates.Let*1, 2, 3, 4, *and*5 *denote the states:A'sgame,B'sgame,Deuce,AdvantageA,andAdvantageB,respectively.Thegame moves from state to state until one player wins.Theprobabilities ofmovingfrom one state to anothercan be summarizedas:*

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & p & q \\ 4 & p & 0 & q & 0 & 0 \\ 5 & 0 & q & p & 0 & 0 \end{array}$$

Coefficient of Variation: The coefficient of variation expresses the sample standard deviation as a percentage of the mean of what is being measured and gives us a measure of relative reliability, and also stability. We now demonstrate this with an example from a tennis game.

Example [6].*At a tennis tournament, Pete Sampras had* 4, 6, 2,5, 1, 3 *and* 2 *double faults in his matches, while Andre Agassi had* 5,2, 6, 4, 6, 4, *and* 3 *double faults. Which player was more consistent inthe sense of number of double faults per match?*

Solution: For Sampras we find that his double fault data has mean and sample standard deviation:

$$\mu_S = \frac{4+6+2+5+1+3+2}{7} = \frac{23}{7} \equiv 3.29$$

$$\sigma_S = \sqrt{\frac{(4-3.29)^2+(6-3.29)^2+(2-3.29)^2+(5-3.29)^2+(1-3.29)^2+(3-3.29)^2+(2-3.29)^2}{7-1}} \equiv 1.8$$

Therefore, a coefficient of variation is: $V_S = \frac{\sigma_S}{\mu_S}\times 100 = \frac{1.8}{3.29}\times 100 \equiv 54.7\%$.

While for Agassi, we see his double fault data has mean and standard deviation: $\mu_A \equiv 4.3,\ \sigma_A \equiv 1.5$, and hence, a coefficient of variation is $V_A = \frac{\sigma_A}{\mu_a} \times 100 = \frac{1.5}{4.3} \times 100 \equiv 34.9\% < 54.7\% = V_S$. Thus, we see that Agassi was more consistent in his double fault pattern.

2. GAME THEORY AND TENNIS

In this section, we will look at games inwhich there is not even one pure-strategy Nash equilibrium—games in whichnone of the players would consistently choose one strategy as that player's equilibriumaction.*A simple example of a game with no equilibrium in pure strategies is thatof a single point in a tennis match*. Game theory assumes that players will calculate and try to maximize their expected payoffs whenprobabilistic mixtures of strategies or outcomes are included.The wordexpected in "expected payoff" is a technical term from probability and statistics. It merely denotesa probability-weighted average. It does not mean this is the payoff that the player should expect inthe sense of regarding it as entitlement.

When players choose to act unsystematically, they pick from among their purestrategies in some random way. The notion of Nash equilibrium also extends easily to include mixed strategies.The game of tennis is one of the oldest sports in the world and considered one of the most mental games out there, considering that in a singles match, two players are pitted against each other through at least two grueling sets. Each player has the chance to *serve*, and therefore each player also has the chance to *receive*.

The question becomes whether we can use **game theory** to predict the behavior of each player on the court. More specifically, the simplest form of these predictive experiments has been to look at the initial *serve* of the player and the initial decision of the returning player, and the outcome of the point (who wins the point). Research has shown that although each player may have a stronger side (forehand(F) or backhand(B)) or able to serve better to one side, it is very rare to find a Pure-Strategy Nash Equilibrium between any two top players, and therefore each player must constantly be adapting his/her**Mixed-Strategy Nash Equilibrium** to fit prior information.*Nash equilibrium is defined as a list of mixed strategies, one for each player, such that the choice of each is his/her best choice, in the sense of yielding the highest expected payoff for his/her, given the mixed strategies of the others.*

First we set up the 2 x 2 matrix as well as state a couple of assumptions.We assume that the only decision for the server and receiver is to go F or B. This assumption is only partially valid, since the server can also decide to serve at the body of the receiver, and can often do this effectively.

Example (Tennis): Consider the following example in a Tennis match

		Receiver F	Receiver B
Server	F	10, 90	70, 30
	B	80, 20	40, 60

The potential pure strategies for the server can be analyzed as follows:

(1). If the server always aims forehands (F) then the receiver (anticipating the forehand serve) will always move forehands and the payoffs will be (10, 90) to server and receiver respectively.

(2). If the server always aims backhands then the receiver (anticipating the backhand serve) will always move backhands and the payoffs will be (40, 60).

However,the server will choose neither (1) nor (2) if he/she wants to win the serve. Thus, the server can increase his/her performance by mixing forehands and backhands.

For example, suppose the server aims forehand with 50% chance and backhands with 50% chance. Then the receiver's payoff is

(1). 0.5*90 + 0.5*20 = 55 if the receiver moves forehands and

(2). 0.5*30 + 0.5*60 = 45 if the receiver moves backhands.

Since it is better to move forehands, the receiver will do that and payoff will be 55. Hence, if the server mixes 50-50 the payoff will be 45, which is already an improvement for the server's payoff.

The next step is to see if we can find a *best mix-strategy* for the *server*. Suppose the server aims forehands with p probability and backhands with (1-p) probability. Then calculating the receiver's payoff, we get:

(1). $p*90 + (1-p)*20 = 20 + 70p$, if the receiver moves forehands and

(2). $p*30 + (1-p)*60 = 60 – 30p$, if the receiver moves backhands.

The receiver will move towards the side that maximize his/her payoff. Therefore, the receiver will move:

(i). forehands if $20 + 70p > 60 – 30p$,

(ii). backhands if $20 + 70p < 60 – 30p$, and

(iii). either one if $20 + 70p = 60 – 30p$.

So the receiver's payoff = $\max\{20 + 70p, 60 – 30p\}$.

In order to maximizing the server's payoff, he/she should minimize the receiver's payoff.The server can do this by setting $20 + 70p$ and $60 – 30p$ equal:

$20 +70p = 60 – 30p$ so that $100p = 40$ and $p = 0.4$.

The server should aim forehands 40% of the time and backhands 60% of the time. Therefore, the receiver's payoff will be 20 + 70*0.4 = 60 - 30*0.4 = 48.

Hence, the server's payoff will be: 100 – 48 =52.

We can use the similar method to analyze the *mix-strategy* for the receiver.

Suppose the receiver moves forehands with probability q. Then the receiver's payoff is:

(1). $q*90 + (1-q)*30 = 30 + 60q$ if the server aims forehands and

(2). $q*20 + (1-q)*60 = 60 – 40q$ if the server aims backhands.

In this situation, the server will aim towards the side that minimizes the receiver's payoff. Hence, the server will aim at:

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(i). forehands if $30 + 60q < 60 - 40q$
(ii). backhands if $30 + 60q > 60 - 40q$
(iii). either one if $30 + 60q = 60 - 40q$.

So the receiver should equate $30 + 60q$ and $60 - 40q$ to maximize the payoff: $30 + 60q = 60 - 40q$ so $100q = 30$ and $q= 0.3$. The receiver should move forehands 30% of the time to maximize the payoff and backhands 70% of time. In this case the receiver's payoff will be:
30 + 60*0.3 = 60 - 40*0.3 = 48. The server's payoff will be 100 - 48 = 52.

After calculating the payoff for both the server and the receiver,the mixed strategy came out to be:
Server: 0.4F + 0.6B
Receiver: 0.3F + 0.7B
And this is the only strategy that cannot be "exploited" by either player.Hence it is a mixed strategy Nash equilibrium.
Important Observation: There is an important observation that found by many mathematicians who study mixed-strategy of tennis: If a player is using a mixed strategy at equilibrium,then he/she should have the same expected payoff from the strategies he/she is mixing. Based on this observation, we can easily find mixed- strategy Nash Equilibrium in a 2 x 2 game.

Example. Let's find the *mixed-strategy Nash Equilibrium* of the following game which has no pure strategy Nash Equilibrium.

		Player 2	
		q L	**(1-*q*)** R
Player 1	*p* U	2, -3	1, 2
	(1-p) D	1, 1	4, -1

Solution. Let *p* be the probability of Player 1 playing U and *q*be the probability of Player 2 playing L at mixed strategy Nash equilibrium. Our objective is finding *p* and *q*.

- AtmixedstrategyNashequilibriumbothplayersshouldhavesameexpectedpayoff sfrom theirtwostrategies.
- ConsiderPlayer1.
  - If Player 1playsUhe/she willreceiveapayoff of2withprobability*q*and1withprobability(1-*q*).Thereforehis/herexpectedpayoff E(U)fromplayingUis$2q+(1-q)$.
  - IfPlayer 1playsDhe/she willreceiveapayoff of1withprobability*q*and4withprobability(1-*q*).Thereforehis/herexpectedpayoff E(D)fromplayingDis$q+4(1-q)$.

Player 1 mixbetweenthetwostrategiesonlyifthesetwoexpectedpayoff s arethesame:
$E(U)=E(D)\Rightarrow 2q+(1-q)=q+4(1-q)\Rightarrow 4q=3\Rightarrow q=3/4$.
ThereforePlayer1willmixbetweenthetwostrategiesonlyifq=3/4.

- Nextlet'sconsiderPlayer2.
  - IfPlayer 2playsLhe/she willreceiveapayoff of-3withprobability*p*and1withprobability(1-*p*), expectedpayoff E(L)fromplayingLis-

3*p*+(1-*p*).

- IfPlayer 2playsRhe/she willreceiveapayoff of2withprobability*p*and -1withprobability(1-*p*).Thereforeherexpectedpayoff E(R)fromplayingRis2*p*+(-1)(1-*p*).

Player 2mixbetweenthetwostrategiesonlyifthesetwoexpectedpayoff saresame:
$E(L)=E(R)\Rightarrow-3p+(1-p)=2p-(1-p)\Rightarrow 7p=2\Rightarrow p=2/7$.
ThereforePlayer2willmixbetweenthetwostrategiesonlyifp=2/7.
Therefore themixed strategyNash equilibriumis:
Player1:Uwithprobability2/7andDwithprobability5/7,
Player2:Lwithprobability3/4andRwithprobability1/4.
WhataboutthemixedNashequilibriumpayoff s?Thepayoff forPlayer1is

$$2\times\frac{3}{4}+1\times\frac{1}{4}=1\times\frac{3}{4}+4\times\frac{1}{4}=\frac{7}{4}$$

and the mixed Nash equilibrium payoff to Player 2 is: $-3\times\frac{2}{7}+1\times\frac{5}{7}=2\times\frac{2}{7}+(-1)\times\frac{5}{7}=\frac{7}{4}$.

3. CALCULUS OF TENNIS

Most players are familiar with the general principle that low tension gives more power and high tension gives more control on the Tennis Racket. The lower tension strings stretch more during impact and thus store more energy. When the ball rebounds from the racket, more energy is returned, so it leaves with a higher speed. The claim that higher string tension gives more control is less easy to explain. There is certainly plenty of anecdotal evidence that players “feel” more control when using a high string tension. Furthermore, in the professional game, players like Andy Roddick and Serena Williams are reported to be using string tensions of over 70 lbs, however recommended tension of string (Tension crosses) is 22-25kg, i.e. 49-55lbs. We discuss this phenomena using Calculus as below:

To compare the average velocity of a Tennis ball struck with Rackets of varying tensions:

Racket 1: (Lower Tension, Main = 45 lbs, Crosses = 40 lbs)
Distance = 74 feet, Time = 1.2 secs.
We get fitting quadratic regression using *Wolfram Mathematica*:

In[1]:=Normal[LinearModelFit[{{0,0},{0.2,6},{0.4,24},{0.6,39},
{0.8,51},{1.0,63},{1.2,74}},***t***^Range[0,2],***t***]]

Out[1]= $-\mathbf{2.83333}+\mathbf{68.75}t-\mathbf{3.27381}t^2$, giving distance, as position of tennis ball at time ***t***

In[2]:= $v_1(t)=\frac{d}{dt}\left(-\mathbf{2.83333}+\mathbf{68.75}t-\mathbf{3.27381}t^2\right)$

Out[2]= $68.75-6.54762t$

In[3]:= $\int_0^{1\cdot 2} v_1(t)dt$

Out[3]= 77.7857

Thus the average velocity of the Tennis ball is obtained on dividing above integral by $(t_1 - t_0) = (1\cdot 2 - 0)$,

In[4]:= 77.7857/1·2

Out[4] = 64.82 feet per second.

<u>Racket 2:</u> (Higher Tension, Main = 55 lbs, Crosses = 53 lbs)
Distance = 62 feet, Time = 1.6 secs, we get again fittingthe quadratic regression:

In[5]:= Normal[LinearModelFit[{{0,0},{0.2,4},{0.4,10},{0.6,20},{0.8,29},
{1.0,38},{1.2,47},{1.4,57},{1.6,62}},***t***^Range[0,2],***t***]]

Out[5]= $\mathbf{-2.17576 + 35.4794}t + \mathbf{3.81494}t^2$

In[6]:= $v_2(t) = \frac{d}{dt}\left(\mathbf{-2.17576 + 35.4794}t + \mathbf{3.81494}t^2\right)$

Out[6]= 35.4794 +7.62987 ***t***

In[7]:= $\int_0^{1\cdot 6} v_2(t)dt$

Out[7]= 66.5333

Thus the average velocity of the Tennis ball is obtained on dividing above integral by (1.6 - 0),

In[8]:= 66.5333/1.6

Out[8]= 41.58feet per second.

Hence, we conclude that the average velocity of the tennis ball struck by the Racket 1 (lower tension) is >the average velocity of the tennis ball struck by the Racket 2 (higher tension). This justifies that the lower tension gives more power, while higher one gives more control.
Similarly, we can also find the average velocity of serve for the player.

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