MULTIPLE INTEGRATION OF A FUNCTION

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ABSTRACT

The multiple integral is a generalization of the definite integral to functions of more than one real variable, for example, f(x, y) or f(x, y, z). Integrals of a function of two variables over a region in R^2 are called double integrals, and integrals of a function of three variables over a region of R^3 are called triple integrals.

Just as the definite integral of a positive function of one variable represents the area of the region between the graph of the function and the x-axis, the double integral of a positive function of two variables represents the volume of the region between the surface defined by the function (on the three-dimensional Cartesian plane where z = f(x, y)) and the plane which contains its domain. (The same volume can be obtained via the triple integral—the integral of a function in three variables—of the constant function f(x, y, z) = 1 over the above-mentioned region between the surface and the plane.) If there are more variables, a multiple integral will yield hypervolumes of multidimensional functions.

Keywords:

Multiple integration, Function, Variable

INTRODUCTION

Multiple integration of a function in n variables: $f(x_1, x_2, \ldots, x_n)$ over a domain D is most commonly represented by nested integral signs in the reverse order of execution (the leftmost integral sign is computed last), followed by the function and integrand arguments in proper order (the integral with respect to the rightmost argument is computed last).

Multiple integrals have many properties common to those of integrals of functions of one variable (linearity, commutativity, monotonicity, and so on). One important property of multiple integrals is that the value of an integral is independent of the order of integrands under certain conditions. This property is popularly known as Fubini's theorem

When the domain of integration is symmetric about the origin with respect to at least one of the variables of integration and the integrand is odd with respect to this variable, the integral is equal to zero, as the integrals over the two halves of the domain have the same absolute value but opposite signs. When the integrand is even with respect to this variable, the integral is equal to twice the integral over one half of the domain, as the integrals over the two halves of the domain are equal.

The integral of order ν of a function $f(r)$ is given by:

$$F_\nu(k) = \int_0^\infty f(r) J_\nu(kr)\, r\, \mathrm{d}r$$

where

J_ν is the Bessel function of the first kind of order ν with $\nu \geq -\frac{1}{2}$

The inverse integralof $F_\nu(k)$ is defined as:

$$f(r) = \int_0^\infty F_\nu(k) J_\nu(kr) k\, \mathrm{d}k$$

which can be readily verified using the orthogonality relationship described below.

Inverting a integralof a function *f*(*r*) is valid at every point at which *f*(*r*) is continuous provided that the function is defined in (0, ∞), is piecewise continuous and of bounded variation in every finite subinterval in (0, ∞), and

$$\int_0^\infty |f(r)|\, r^{\frac{1}{2}}\,\mathrm{d}r < \infty.$$

However, like the integral, the domain can be extended by a density argument to include some functions whose above integral is not finite, for example

$$f(r) = (1+r)^{-3/2}.$$

An alternative definition says that the integralof *g*(*r*) is:

$$h_\nu(k) = \int_0^\infty g(r) J_\nu(kr)\,\sqrt{kr}\,\mathrm{d}r$$

The two definitions are related:

$$\text{If } g(r) = f(r)\sqrt{r} \text{ then } h_\nu(k) = F_\nu(k)\sqrt{k}.$$

This means that, as with the previous definition, the integraldefined this way is also its own inverse:

$$g(r) = \int_0^\infty h_\nu(k) J_\nu(kr)\,\sqrt{kr}\,\mathrm{d}k$$

The obvious domain now has the condition

$$\int_0^\infty |g(r)|\,\mathrm{d}r < \infty$$

but this can be extended.

According to the reference given above, we can take the integral as the limit as the upper limit goes to infinity (an improper integral rather than aLebesgue integral) and in this way the integraland its inverse work for all functions in $L^2(0, \infty)$.

The Bessel functions form an orthogonal basis with respect to the weighting factor *r*:

$$\int_0^\infty J_\nu(kr)J_\nu(k'r)r\,\mathrm{d}r = \frac{\delta(k-k')}{k}, \qquad k,k'>0.$$

RESEARH STUDY

If $f(r)$ and $g(r)$ are such that their integral$F_\nu(k)$ and $G_\nu(k)$ are well defined, then the Plancherel theorem states

$$\int_0^\infty f(r)g(r)r\,\mathrm{d}r = \int_0^\infty F_\nu(k)G_\nu(k)k\,\mathrm{d}k.$$

Parseval's theorem, which states:

$$\int_0^\infty |f(r)|^2 r\,\mathrm{d}r = \int_0^\infty |F_\nu(k)|^2 k\,\mathrm{d}k,$$

is a special case of the Plancherel theorem. These theorems can be proven using the orthogonality property.

The integralof order zero is essentially the 2-dimensional Fourier transform of a circularly symmetric function.

Consider a 2-dimensional function $f(\mathbf{r})$ of the radius vector **r**. Its Fourier transform is:

$$F(\mathbf{k}) = \iint f(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}\,\mathrm{d}\mathbf{r}.$$

With no loss of generality, we can pick a polar coordinate system (r, θ) such that the **k** vector lies on the $\theta = 0$ axis(in K-space). The Fourier transform is now written in these polar coordinates as:

$$F(\mathbf{k}) = \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} f(r,\theta) e^{ikr\cos(\theta)}\, r\, d\theta\, dr$$

where θ is the angle between the **k** and **r** vectors. If the function f happens to be circularly symmetric, it will have no dependence on the angular variable θ and may be written $f(r)$. The integration over θ may be carried out, and the Fourier transform is now written:

$$F(\mathbf{k}) = F(k) = 2\pi \int_0^{\infty} f(r) J_0(kr) r\, dr$$

which is just 2π times the zero-order integralof $f(r)$. For the reverse transform,

$$f(\mathbf{r}) = \frac{1}{(2\pi)^2} \iint F(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{k} = \frac{1}{2\pi} \int_0^{\infty} F(k) J_0(kr) k\, dk$$

so $f(r)$ is $1/2\pi$ times the zero-order integralof $F(k)$.

SIGNIFICANCE OF THE STUDY

For an n-dimensional Fourier transform,

$$F(\mathbf{k}) = \int f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^n\mathbf{r}$$

if the function f is radially symmetric, then:

$$k^{n/2-1}F(k) = (2\pi)^{n/2}\int_0^\infty r^{n/2-1}f(r)J_{n/2-1}(kr)r\,dr$$

To generalize: If f can be expanded in a multipole series,

$$f(r,\theta) = \sum_{m=-\infty}^{\infty} f_m(r)e^{im\theta},$$

and if θ_k is the angle between the direction of **k** and the $\theta = 0$ axis,

$$\begin{aligned}
F(\mathbf{k}) &= \int_0^\infty r\,\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\, f(r,\theta)e^{ikr\cos(\theta-\theta_k)}\\
&= \sum_m \int_0^\infty r\,\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\, f_m(r)e^{im\theta}e^{ikr\cos(\theta-\theta_k)}\\
&= \sum_m e^{im\theta_k}\int_0^\infty r\,\mathrm{d}r\, f_m(r)\int_0^{2\pi}\mathrm{d}\varphi\, e^{im\varphi}e^{ikr\cos\varphi} \quad \varphi = \theta-\theta_k\\
&= \sum_m e^{im\theta_k}\int_0^\infty r\,\mathrm{d}r\, f_m(r)2\pi i^m J_m(kr)\\
&= 2\pi\sum_m i^m e^{im\theta_k}\int_0^\infty f_m(r)J_m(kr)r\,\mathrm{d}r.\\
&= 2\pi\sum_m i^m e^{im\theta_k}F_m(k)
\end{aligned}$$

where $F_m(k)$ is the m-th order Hankel transform of $f_m(r)$.

Additionally, if f_m is sufficiently smooth near the origin and zero outside a radius R, it may be expanded into a Chebyshev series,

$$f_m(r) = r^m \sum_{t\geq 0} f_{mt} \left(1 - \left(\tfrac{r}{R}\right)^2\right)^t, \qquad 0 \leq r \leq R.$$

so that

$$\begin{aligned} F(\mathbf{k}) &= 2\pi \sum_m i^m e^{im\theta_k} \sum_t f_{mt} \int_0^R r^m \left(1 - \left(\tfrac{r}{R}\right)^2\right)^t J_m(kr) r \, \mathrm{d}r && (*) \\ &= 2\pi \sum_m i^m e^{im\theta_k} R^{m+2} \sum_t f_{mt} \int_0^1 x^m (1-x^2)^t J_m(kxR) x \, \mathrm{d}x && x = \tfrac{r}{R} \\ &= 2\pi \sum_m i^m e^{im\theta_k} R^{m+2} \sum_t f_{mt} \frac{t! 2^t}{(kR)^{1+t}} J_{m+t+1}(kR). \end{aligned}$$

The above can be viewed as a more general case that is not as constrained as the previous case in the previous section. The numerically important aspect is that the expansion coefficients f_{mt} are accessible with Discrete Fourier transform techniques. Insertion into the previous formula yields.

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