International Journal in Physical and Applied Sciences
Volume 06 Issue 05, May 2019 ISSN: 2394-5710 Impact Factor: 4.657
Journal Homepage: http://ijmr.net.in, Email: irjmss@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal

IMPACT OF FIXED POINT THEORY IN METRIC SPACE

Manish Pithadia
Research Scholar

Abstract
Several issues in pure and applied mathematics have the fixed point of some mapping F as their solutions. Therefore, a number of numerical analysis and approximation theory procedures involve successive approximations to the fixed point of an approximate mapping to be obtained. In this paper, we also established our objective to address fixed point theory and its applications in metric spaces.

Keywords: Fixed-point theory, Metric space, Complete Metric space, con-tinuous function

Introduction

The well-known Banach [1] contraction principal states that "If X is complete metric space and f is a contraction mapping on X into itself,then f has unique fixed point in X". Many mathematicians worked on this principal. Kanan[4] proved that "If T is self-mapping of a complete metricspace X into itself satisfying :

Let f and g be self-mappings of a metric space

(X, d). The mappings f and g are said to be **compatible** if$\lim_{n\to\infty} d(gfx_n, fx_n) = 0$, whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

In 1998, Liu, Xu and Cho[64] proved the following theorem

Main Result

Theorem 1. Let f be continuous self-mapping of a complete metric space (X, d). Then following are equivalent:

i) f has a fixed point in X.

ii) there exists z ∈X, a mapping g: X →X and functions ϕ from X into

[0,∞) such that f and g are compatible, g(X) ∈ f(X), g is continuous and (*)$d(gx,gy) \le r\, d(fx,z)+[\phi(fx)-\phi(gx)]$

for all x,y ∈ X and some r ∈ [0,1)

Now we prove the following theorem:

Theorem 2. Let f and S be continuous self-mappings of a complete metric space (X, d). Then following are equivalent:

(1.1) f and S have a common fixed point.

(1.2) there exists a mapping g : X →X and functions ϕ, ψ from X into[0,∞) such that pairs {f, g} and {S, g} are compatible, g(X)⊆f(X),
g(X)⊆S(X), g is continuous and

($d(gx,gy) \leq a_1 d(fx,Sy) + a_2\, d(Sx,fy) + a_3\, d(fx,gx) + a_4 d(fy,gy)$
$+ a_5\, d(Sx,gx) + a_6\, d(Sy,gy) + a_7\, d(fx,gy) + a_8\, d(fy,gx)$
$+ a_9 d(Sx,gy) + a_{10} d(Sy,gx) + [\phi(fx) - \phi(gx)]$
$+ [\psi(Sy) - \psi(gy)]$ for all x, y $\in X$, with
$a_1, a_2, \ldots, a_{10}$ are in [0, 1) where $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + 2a_7 + 2a_9 < 1$
and$a_1 + a_2 + a_7 + a_8 + a_9 + a_{10} < 1$.

Proof: For (1.1) ⇒ (1.2)

Suppose w is the fixed point of f and S then fw = w and Sw = w.
Take gx = w for all x ∈ X. Let ϕ and ψ be constant.Also,

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = d(fw, w) = d(w, w) = 0$$

and

$$\lim_{n\to\infty} d(Sgx_n, gSx_n) = d(Sw, w) = d(w, w) = 0.$$

Hence compatibility follows. Thus, (2.2.1) ⇒ (2.2.2).
Now, for (2.2.2) ⇒ (2.2.1). Let $x_0 \in X$ be an arbitrary point of X. Since $g(X) \subseteq f(X)$, $g(X) \subseteq S(X)$, so we construct a sequence $\{x_n\}_{n=1}^{\infty}$, in X by
$gx_{n-t} = fx_n = Sx_n$ for $n \geq 1$. From (**) it follows that

$$d_{j+1} = d(gx_j, gx_{j+1}) \leq a_1 d(fx_j, Sx_{j+1}) + a_2 d(Sx_j, fx_{j+1}) + a_3 d(fx_j, gxj)$$

$$+ a_4 d(fx_{j+1}, gx_{j+1}) + a_5 d(Sx_j, gx_j) + a_6 d(Sx_{j+1}, gx_{j+1})$$

$$+ a_7 d(fx_j, gx_{j+1}) + a_8 d(fx_{j+1}, gx_j) + a_9 d(Sx_j, gx_{j+1})$$

$$+ a_{10} d(Sx_{j+1}, gx_j) + [\phi(fx_j) - \phi(gx_j)]$$
$$+ [\psi(Sx_{j+1}) - \psi(gx_{j+1})]$$

$$= a_1 d(fx_j, fx_{j+1}) + a_2\, d(fx_j, fx_{j+1}) + a_3\, d(fx_j, fx_{j+1})$$

$$+ a_4 d(fx_{j+1}, fx_{j+2}) + a_5 d(fx_j, fx_{j+1}) + a_6 d(fx_{j+1}, fx_{j+2})$$

$$+ a_7 d(fx_j, fx_{j+2}) + a_8 d(fx_{j+1}, fx_{j+1}) + a_9 d(fx_j, fx_{j+2})$$

$$+ a_{10} d(fx_{j+1}, fx_{j+1}) + [\phi(fx_j) - \phi(fx_{j+1})]$$

$$+ [\psi(fx_{j+1}) - \psi(fx_{j+2})]$$

Put $d_n = d(fx_n, fx_{n+1})$ for $n \geq 0$.

International Journal in Physical and Applied Sciences
Volume 06 Issue 05, May 2019 ISSN: 2394-5710 Impact Factor: 4.657
Journal Homepage: http://ijmr.net.in, Email: irjmss@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal

$d_{j+1} \le a_1 d_j + a_2 d_j + a_3 d_j + a_4 d_{j+1} + a_5 d_j + a_6 d_{j+1} + a_7 d_j + a_7 d_{j+1} + a_9 d_j + a_9 d_{j+1} + [\phi(fx_j) - \phi(fx_{j+1})] + [\psi(fx_{j+1}) - \psi(fx_{j+2})]$

$d_{j+1} = d(gx_j, gx_{j+1}) \le a d_j + b[\phi(fx_j) - \phi(fx_{j+1})] + b[\psi(fx_{j+1}) - \psi(fx_{j+2})]$

Where

$$a = \frac{a_1 + a_2 + a_7 + a_8 + a_9}{1 - a_4 - a_6 - a_7 - a_9} \quad and \quad b = \frac{1}{1 - a_4 - a_6 - a_7 - a_9}$$

On adding above inequality from j = 0 to j = n

$$\sum_{j=0}^{n} d_{j+1} \le a \sum_{j=0}^{n} d_j + b \sum_{j=0}^{n} [\phi(fx_j) - \phi(fx_{j+1})] + b \sum_{j=0}^{n} [\Psi(fx_{j+1}) - \Psi(fx_{j+2})]$$

Since $d(x_i,y_i) \ge 0$ and $0 \le a \le 1$, we get

$$\sum_{j=0}^{n} d_{j+1} \le \frac{a}{1-a} d_0 + \frac{b}{1-a} [\phi(fx_0) - \phi(fx_{n+1})] + \frac{b}{1-a} [\Psi(fx_1) - \Psi(fx_{n+2})]$$

Therefore, the series $\sum_{n=1}^{\infty} d_n$ is convergent. For any n, p ≥ 1, we

have by triangle inequality

$$d(fx_n, fx_{n+p}) \ge \sum_{i=n}^{n+p-1} d_i$$

This implies that $\{fx_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since X is complete, so there exists a point t ∈ X such that $\lim_{n\to\infty} fx_n = t$. But f, S and g are continuous and pairs f, g and S, g are compatible, hence

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0 \Rightarrow d(ft, gt) = 0 \text{ as } n \to \infty, \text{i.e.}, ft = gt$$

$$\lim_{n\to\infty} d(gSx_n, Sgx_n) = 0 \Rightarrow d(gt, St) = 0 \text{ as } n \to \infty, \text{i.e.}, gt = St$$

International Journal in Physical and Applied Sciences
Volume 06 Issue 05, May 2019 ISSN: 2394-5710 Impact Factor: 4.657
Journal Homepage: http://ijmr.net.in, Email: irjmss@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal

Thus ft = gt = St.

Now from (**),

$d(ft,fx_{j+1})=d(gt,gx_j)\leq(a_1 + a_2 + a_8 + a_{10})d(fx_j, ft)+(a_4+a_6)d(fx_j,fx_{j+1})$

$+ (a_7+a_9)d(ft, fx_{j+1}) +[\psi(fx_{j+1}) - \psi(fx_{j+2})]$

On adding above inequality for j = 0 to j = n, we obtain

$$\sum_{j=o}^{n} d(ft, fx_{j+1}) \leq (a_1 + a_2 + a_8 + a_{10}) \sum_{j=0}^{n} d(fx_j, ft)$$

$$+(a_4 + a_6)\sum_{j=0}^{n} d\left(fx_j, fx_{j+1}\right) + (a_7 + a_9)\sum_{j=0}^{n} d(ft, fx_{j+1})$$

$$+ \sum_{j=0}^{n}[\Psi(fx_{j+1}) - \Psi(fx_{j+2})]$$

$$(1 - a_1 - a_2 - a_7 - a_8 - a_9 - a_{10}) \sum_{j=o}^{n} d\left(ft, fx_{j+1}\right) \leq (a_1 + a_2 + a_8 + a_{10}) d(fx_0, ft)$$

$$+(a_4 + a_6)\sum_{j=0}^{n} d\left(fx_j, fx_{j+1}\right)$$

$$+ \sum_{j=0}^{n}[\Psi(fx_{j+1}) - \Psi(fx_{j+2})]$$

Or

$$\sum_{j=0}^{n} d(ft, fx_{j+1}) \leq c\, d(fx_0, ft) + d\sum_{j=0}^{n} d\left(fx_j, fx_{j+1}\right) + e[\psi(fx_1) - \psi(fx_{n+2})]$$

Where

$$c = \frac{a_1 + a_2 + a_8 + a_{10}}{1 - a_1 - a_2 - a_7 - a_8 - a_9 - a_{10}} \quad and \quad d = \frac{a_4 + a_6}{1 - a_1 - a_2 - a_7 - a_8 - a_9 - a_{10}}$$

$$e = \frac{1}{1 - a_1 - a_2 - a_7 - a_8 - a_9 - a_{10}}$$

Since the series $\sum_{n=1}^{\infty} d(fx_n, fx_{n+1})$ is convergent and
$a_1 + a_2 + a_7 + a_8 + a_9 + a_{10} < 1$,it follows that the series
$\sum_{n=1}^{\infty} d(ft, fx_n)$ is also convergent. This implies that
$fx_n \to ft$ as $n \to \infty$, i.e., $t = ft = gt = St$.
This completes the proof of the theorem.

Above Theoremextends, improvesand unifies the
Theorem of Jungck [48], Theorem 2 of Fisher [36] and the following
Theorem 3.3 of Liu, Xu and Cho [64].

REFERENCES

[1] FrechetM.Surquelquespointsducalculfonctionnel.Rend.Circ.Mat.Palermo. 1906; 22: 1 – 72.

[2] Shirali S. Metric space. London Springer;2006.

[3] LipschutzsS.Schaumsoutlineoftheoryandproblemsofgeneraltopology.New York: Schaum;1965.

[4] Banach S. Sur les operations dans les ensembles abstraits et leurapplication aux equations integrals, Fundamenta Mathematicae, 1922; 3: 133 –181.

[5] BonsallFF.Lecturesonsomefixedpointtheoremsoffunctionalanalysis.Tata Institute of Fundamental Research: Bombay,1962.

[6] Caccioppoli R. Sugli elementi uniti delle transformazioin funzionali. Rand. Sam. Math. Padova. 1932; 3: 1 –15.

[7] Rakotch E. A note on contractive mappings. Proc. Amer. Math. Soc. 1962; 13:459 – 465.

[8] EdelsteinM.AnextensionofBanach'scontractionprinciple.Proc.Amer.Math. Soc. 1962; 12:7 –10.

[9] Sehgal VM. Some fixed point theorems in function analysis and probability. Ph.D dissertation Wayne State Univ.: Michigan;1966.

[10] Wong JSW. Two extensions of the Banach contraction mapping principles. JournalofMathematicalAnalysisandApplications.1968;22:438–443.

[11] Kannan R. Some result on fixed points. Bull Calcutta Math. Soc. 1968; 60:71– 76.

[12] ReichS.Someremarksconcerningcontractionmappings.CanadMath.Bull. 1971; 14: 121–124.

[13] Chatterjee SK. Fixed point theorem. C. R. Acad. Bulgare Sci. 1972; 25: 727 - 730.

[14] BianchiniRMT.SunProblemadiS.Reichriguardantelateoriadeipuntifissi. Boll.Un.Mat.Ital.1972;5:103–108.

[15] Sehgal VM. On fixed and periodic points for a class of mappings. J. London Math. Soc. 1972; 5: 571 –576.

[16] CaristiJ.Fixedpointtheoremsformappingssatisfyinginwardnessconditions, Transactions of the American Mathematical Society. 1976; 215: 241 –251.

[17] JaggiDS.Someuniquefixedpointtheorems.IndianJ.PureAppl.Math.1977; 8: 223 –230.

[18] Rhoades BE. Contractive definition revisited. Contemporary Mathematics. 1983; 21: 189 –205.

[19] Rhoades BE. A comparisons of various definitions of contractive mappings. Trans. Amer. Math. Soc. 1977; 226: 257 –290.

[20] BerindeV.ApproximationfixedpointsofweakcontractionsusingthePicard iteration. Nonlinear Anal Forum. 2004; 9(1):