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Generalization of Convex Functions

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Abstract

In the following paper, we undertake a broadening of the conventional notion of convex functions by introducing a novel concept: S-convex functions. Our objective is to delve into the properties of S-convex functions as they relate to the epigraphs of these functions. Additionally, we explore several illustrative examples of S-convex functions.

Keywords: Convex set, S-convex set, convex function.

1. Introduction

Convex functions form a cornerstone of mathematical analysis and optimization theory, with a rich history dating back to the early 18th century. The study of convexity and convex functions has not only deepened our understanding of mathematical principles but has also offered valuable insights into real-world problem-solving. We embark on an exploration of convex functions, their intrinsic properties, and their significance in diverse academic and practical domains.

A convex function, by definition, exhibits a particular structural property: the line segment between any two points on its graph lies either above or on the graph itself. In simpler terms, the function's curvature is consistently directed upwards. This fundamental property has far-reaching implications, as it leads to a variety of desirable mathematical properties that make convex functions an indispensable tool in optimization, economics, and engineering, among others.

Various generalized notions of convexity have established their significance as pivotal elements in optimization theory, spanning across domains such as Minkowski spaces, convex analysis, axiomatic convexity, and even combinatorial theory [3-5, 13, 14]. In the pursuit of widening the applicability of their findings to more extensive optimization scenarios, these concepts have undergone multiple extensions and diversifications, employing innovative and novel techniques. Notably, in 1981, Hanson [10] introduced the concept of Invexity, a profound notion that expanded upon the classical concept of convexity.

The process of generalizing convexity is not a straightforward endeavor. Many attempts have resulted in certain shortcomings, with some generalizations merely appearing as formal extensions of the core concept of convexity [2, 8, 17, 25, 26, 28]. However, in contrast, there are extensions like geodesic convexity that have demonstrated exceptional significance in various contexts.

Youness [27] introduced a fresh paradigm by defining a distinct class of sets and functions, termed E-convex sets, and E-convex functions, respectively, by relaxing the conventional definitions of convex sets and functions. The pioneering work of Youness [27] has served as a catalyst, inspiring a multitude of subsequent research that has significantly broadened the scope of E-convexity in optimization theory [6, 7, 9, 21, 24]. Nonetheless, it is imperative to acknowledge that Yang [21] identified inaccuracies in certain results proposed by Youness [27]. Subsequently, Syau and Lee [21] introduced the notion of E-quasiconvex functions and explored the properties of both E-convex and E-quasiconvex functions.

Beyond the Euclidean space, Rapcsak [19] and Udriste [22] proposed a compelling generalization of convexity within the framework of Riemannian manifolds. In this context, the linear space is replaced by the Riemannian

manifold, and the traditional line segment is substituted with a geodesic. Rapcsak [19] extended this concept to derive effective optimality conditions, further applying these results to complementarity problems. Additionally, Pini [18] introduced the concept of Riemannian Invexity, while Mititelu [15] conducted a comprehensive investigation into its generalization.

Numerous research articles have made significant contributions to the field of convex functions, exemplified by references [12] and [16]. Furthermore, these studies have introduced novel generalizations of convex functions, as evidenced by references [20], [23] and [29], and have also presented fresh insights into inequalities and determinants.Building upon the foundation laid by previous research endeavors [1,9, 11, 21, 24, 27] and recognizing the pivotal role of convexity and its generalized forms, this paper aims to contribute to the ongoing exploration and understanding of these essential concepts.

Our research is underpinned by the belief that a deep comprehension of convex functions is essential for researchers, practitioners, and students alike, as they offer elegant solutions to intricate problems. This paper serves as a roadmap for navigating the intricate landscape of convexity, highlighting its importance as a mathematical concept and practical tool. In the current note, we defined the S-convex functions for S-convex sets and studied its properties. In the next section, we give the basis definition and results required for our further study.

2. Basic definitions

Definition: (Convex set) Let A be a non-empty subset of a linear space E.For $x, y \in S$ and $\lambda, \mu \geq 0$, then$A$ is called a Convex Set whenever,

$$\lambda x + \mu y \in A \text{ for } \lambda + \mu = 1.$$

Definition: (S-Convex set) Let A be a non-empty subset of a linear space E. Now, for every $x, y \in A$ and $\lambda, \mu \geq 0$ be (scalars), we shall say that A is a S-convex Set, if $\lambda x + \mu y \in A$ for $\lambda + \mu \leq 1$. Thus, it is observed that, every S-convex Set is a convex Set, but the converse may or may not be true. In this way a S-convex set is more general than a convex Set. Equivalently a convex Set is a particular case of S-convex Set.

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Definition: (Convex function) A function defined on a convex set A is said to be convex function if it possesses the following property:
for given $x, y \in A$ and $t \in [0,1]$, we get:

$$f(tx + (1-t)y) \leq t\,f(x) + (1-t)\,f(y).$$

Definition: (Graph of a function)
For any f: X → Y, the graph of a function is defined to be the set:
Gr(f)={(x,y): x ∈ X, y ∈ Y and y=f(x)}.
Definition: (Epigraph of f) Given a function $f: A \to R, with\ A \subset X$, we define its epigraph as:

$$epi(f) = \{(x,r): x \in A, y \in R, \quad f(x) \leq r\}.$$

If we extend the function by defining the $\tilde{f}(x) = \infty$, if $x \notin A$, then we have
$\mathrm{epi(f)} = \mathrm{epi}(\tilde{\mathrm{f}})$.
Therefore, f is convex iff $\tilde{f}$ is convex.
Let us illustrate some nice results of convex functions.
Theorem: (Global Minimum): A convex function f(x) has a unique global minimum, which is also a local minimum.
Theorem: A function $f: X \to R$ is convex iff its epigraph is convex.
Definition: (Metric function)Let *M*be a non-empty set then a real valued function *d* defined on *M* × *M* is called a distance function (or metric function or simply metric on *M*) if the following conditions are satisfied:

i) $d(x, y) \geq 0$

ii) $d(x, y) = 0 \Leftrightarrow x = y$

iii) $d(x, y) = d(y, x)$

iv) $d(x, z) \leq d(x, y) + d(y, z)$

Here the condition (iii) is known as the condition of symmetry and condition (iv) is known as triangle inequality.

Here (*x*, *y*) is called the distance between *x* and *y*.

Also, d (*x*, *y*), due to symmetry, does not depend on the order of the element.

Let M be a metric space with metric d. Let x∈X and let A be a subset of M and define:
$\mathrm{d(x, A)} = d_A(\mathrm{x}) = \inf\{\mathrm{d(x, a)};\ \mathrm{a} \in \mathrm{A}\}$.

3. Main Results

In the present section, firstly we will define S-convex functions.
Definition: (S-convex function) A function f: $R^n \to R$ is said to be S-convex on a set $M \subset R^n$ iff
$f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y)$ such that $0 \leq \lambda + \mu \leq 1$.
In case $f(\lambda x + \mu y) \geq \lambda f(x) + \mu f(y)$, the function is called S-concave function.
We can observe that for a S-convex function, the domain is necessarily a S-convex set, otherwise, we are able to get any two points x, y in the domain of f such that f(x), f(y) ∈ R and a point z ∈ [x, y] with f(z) =∞.

If the sign in the above inequalities are strict, we say f is strictly S-convex function or strictly S-concave function respectively.

Remark: Every S-convex map is convex, but the converse may or may not be true. This gives us that S-convex function is the generalization of convex function.

Examples: The norm function defined on X to R is S-convex function. Also, we can see that indicator function δ_A is also S-convex function.

Definition:(S-convex set in $R^n \times R$): A set V ⊂ $R^n \times R$ is said to be S-convex if (x, α), (y, β) ∈ V, we have:

$$(\lambda x+ \mu y, \lambda\alpha+ \mu \beta) \in V \text{ for } 0\leq \lambda+ \mu\leq 1.$$

Theorem: A function f defined on a S-convex set $A \subset R^n$ is S-convex on A iff epi(f) is S-convex in $R^n \times R$.

Proof: Assume epi(f) is S-convex.

We need to prove that f is also S-convex.

Let x, y ∈ A, then

(x, f(x)), (y, f(y)) ∈ epi(f)

As epi(f) is S-convex, it gives us:

λ (x, f(x))+ μ (y, f(y)) ∈ epi(f)

⇒(λ x, λ f(x))+ (μ y, μ f(y)) ∈ epi(f)

⇒(λ x+ μ y, λ f(x)+ μ f(y)) ∈ epi(f)

⇒f (λ x + μ y) ≤ λ f(x)+ μ f(y)).

Thus, we conclude that f is a S-convex function.

Converse:Let f is a S-convex function.

Now, to prove is that epi(f) is S-convex function.

Let (x, f(x)), (y, f(y)) ∈ epi(f)

As x, y ∈ A and A is S-convex set. This gives us,

λx+ μy ∈ A, where 0≤ λ+ μ≤1.

Now as f(x), f(y) ∈ R and R is S-convex set.

We can write for 0≤ λ+ μ≤1,

λ f(x)+ μ f(y) ∈ R.

By using the S-convexity of the function f, we can write.

f (λ x+ μ y) ≤λ f(x)+ μ f(y)

(λ x+ μ y, λ f(x)+ μ f(y)) ∈ epi(f).

Hence, epi(f) is a S-convex set.

Theorem: Assume that $(f_i)_{i\in I}$is a family of S-convex and bounded above functions on a set$A \subset R^n$. Then, the function $f = \sup_{i \in I}(f_i(x))$is S-convex function on A, where I is the index set.

Proof: Consider each f_i is S-convex function on A.

Therefore, epi(f_i) is S-convex for each i ϵ I with the help of above theorem.

That means,

epi(f_i)= {(x,α): x ∈ X, α ∈ R, f_i(x) ≤ α} is S-convex set in $R^n \times R$.

We know that the intersection of S-convex sets is again a S-convex set. Therefore, we get:

$\cap$ epi(f_i) = {(x, α), x∈ X, α ∈ R, $f_i(x) \leq \alpha$, I ϵ I}
={(x, α): x∈ X, α ∈ R, f(x) ≤ α},
where f(x)=sup $f_i(x)$.
Thus epi(f) is S-convex function.
Therefore, by above theorem, we get f is S-convex function.
Lemma: Let $f, g: X \to (-\infty, \infty]$ be two functions. We define.

$$h(x) = \inf_{w \in X}\big(f(w) + g(x - w)\big),$$

then $epi(h) = epi(f) + epi(g)$ if the infimum attains always.
Proof: If (x, r) ϵ epi(f) and (y, s) ϵ epi(g), we have
r≥ f(x), s ≥ f(y), this gives us.
r+ s ≥ f(x) +g(y) = f(x) +g(y-x +x) ≥ h(x+ y),
hence (x+ y, r+ s) ∈ epi(h).
Converse: Consider (z, γ) ∈ epi(h), we have
γ ≥ h(z)= f(w)+g(z-w) for some w ϵ X.
Now, (z, γ)=(w, f(w))+ (z-w, γ- f(w)),
where (w, f(w)) ∈ epi(f) and (z-w, γ- f(w)) ∈ epi(g).
We shall designate the function h, which we defined in the preceding lemma, as in convolution (f # g). Therefore, as stated in the Lemma, the sum of the epigraphs of the two functions is equivalent to the epigraph of the inf-convolution. It is possible to observe that the attainment of the inf is not a prerequisite for the inclusion epi(f) + epi(g)⊂ epi (f #g).
Theorem: Let $A \subset R^n$, then d_A is S-convex if A is a closed S-convex set.
Proof: Consider $d_A = \inf_{z \in A}|x - z| = \inf_{y \in R^n}(\gamma_A + |x - y|)$. We may use above Lemma since the infimum will attain because A is closed. Therefore, epi(d_A)=epi(γ_A)+epi (|.|).Thus, we conclude that epi(d_A) is S-convex, which gives us that $d_A$is S-convex.
Theorem: Let $x_0 \subset X$, r >0, and f: B(x_0, r)→R be a S-convex function. Assume that |f(x)|≤ M for every x ϵ B(x_0, r). Then |f(x)-f(y)| $\leq \frac{4M}{r}|x - y|$ for every $x, y \in B\left(x_0, \frac{r}{2}\right)$.
Proof: Let x, y be two different points of $B\left(x_0, \frac{r}{2}\right)$, let us define

$$z = y + \frac{r}{2|y - x|}(y - x),$$

We can check that $z \in B(x_0, r)$.
Moreover, $2|y - x|z = 2|y - x|y + r(y - x) = (2|y - x| + r)y - rx$, and

$$y = \frac{2|y - x|}{2|y - x| + r}z + \frac{r}{2|y - x| + r}x.$$

As f is S-convex function, we get:

$$f(y) \leq \frac{2|y - x|}{2|y - x| + r}f(z) + \frac{r}{2|y - x| + r}f(x)$$
$$= \frac{2|y - x|}{2|y - x| + r}\big(f(z) - f(x)\big) + f(x).$$

This can be written as:

$$
\begin{aligned}
f(y)-f(x) &\le \frac{2\,|\,y-x\,|}{2\,|\,y-x\,|+r}\big(f(z)-f(x)\big) \\
&\le \frac{2\,|\,y-x\,|}{2\,|\,y-x\,|+r}\,|\,f(z)-f(x)\,| \\
&\le \frac{2\,|\,y-x\,|}{2\,|\,y-x\,|+r}\,2\,M \\
&\le \frac{2\,|\,y-x\,|}{r}\,2\mathrm{M} = \frac{4\,M}{r}|y-x|.
\end{aligned}
$$

Hence, the proof.

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