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SOME EXACT SOLUTIONS OF EINSTEIN'S FIELD EQUATIONS FOR ANISOTROPIC FLUID SPHERE

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ABSTRACT :

In this paper we have obtained exact analytical solutions of Einstein's field equations in two cases for static anisotropic fluid sphere by assuming that space time is conformally flat. In first case a suitable form of energy density ρ has been taken and in second model a judicious choice of metri function g_{11} has been used. Both the models are physically realistic and free from singularities. We have also calculated energy density ρ, radial and tangential pressures for both the models. It is found that densities for these models drop continuously from their maximum values at the centre to the values which are +ve at the boundary

Key Words : Pressure, density, anisotropic, fluid sphere, conformal

1. Introduction

Perfect fluid spheres with homogeneous density and isotropic pressure in general relativity were considered by Schwarzechild [20] and the solutions of relativistic field equations were obtained. Tolman [24] developed a mathematical method for solving Einstein's field equations applied to static fluid spheres in such a manner as to provide explicit solutions in terms of known analytic functions. A number of new solutions were thus obtained and the properties of three of them were examined in detail. These solution were used by Oppenheiner and Volkoff (19) in the study of massive neutron cores. Mehra et. al. [18] have obtained a general solution of the field equations for a composite sphere having a number of shells of different densities. Durgapal and Gehlot [6] have obtained exact internal solutions for dense massive stars in which the central pressure and density are infinitely large. Durgapal and Gehlot [7,8] have further obtained exact solutions for a massive sphere with two different density distribution. The density being minimum at the surface varies inversely as the square of the distance from the centre. The distribution has a core of constant density and radius. Static and non-static solutions of Einstein's field equations have also been extensively discussed by Leibovitz ([15(a))], [15(b)] for the spherical distribution. Solutions of Einstain's field equations for perfect fluid sphere have been also studied by Adler [1], Whitman [25] Singh and Yadav [20(a)] and Yadav and Saini [26].

In investigations concerning massive objects in general relativity the matter distribution is usually assumed to be locally isotopic. However, in the last few years theoretical studies on realistic steller models indicate that some massive objects may be locally anisotropc [1, 4, 17]. There are a number of interesting solutions that have provided insight into the effects of anisotropy on star parameters [5, 12, 14]. However, many of these solutions have a limited applicability to astrophysical situations since they do not satisfy certain physical restrictions usually imposed upon density and pressure, viz. that the pressures should not exceed the energy density (dominant energy condition), and that the (adiabatic) derivatives of the pressure with respect to the density should be less than or equal to unity [11] (macrocausality condition).

Exact analytical solutions of Einstein's field equagtions are of much value in general relativity. These solutions are generally obtained by using different conditions and assumptions. One of the assumptions made for obtaining the solutions is that the space time be conformally flat. This assumption has been widely used in relativity. Theory [2, 3, 10, 16, 21]. For some other workers in this field see references [1(a), 5(a), 18(a), 19(a), 26, 27,].

Here in this paper we have obtained two exact analytical solutions of Einstein's field equations for static anisotropic fluid spheres by assuming that space time is conformally flat. In first case we have used a judicious choice of energy density ρ and in the second model we have chosen a suitable form of metric potential g_{11}. Both the models are physically reasonable and free from singularities. Energy density ρ, radual and tangential pressures have been calculated for both the models. It is seen that densities for these models drop continuously from their maximum values at the centre to the values which are positive at the boundary.

2. The Field Equations and Their Solution

We take the line element in the form

(2.1) $$ds^2 = e^{\beta}dt^2 - e^{\alpha}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where α and β are functions of r only. The theory of spherically symmetric space-times has also been discussed in good detail by Takeno [23]. The Einstein's field equations in general relativity

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(2.2) $R^i_j - \frac{1}{2} R\delta^i_j = -8\pi T^i_j$

For the metric (2.2.1) are (Tolman [24])]

(2.3) $-8\pi T^1_1 - e^{-\alpha}\left[\beta'/r + 1/r^2\right] - 1/r^2$

(2.4) $-8\pi T^2_2 = -8\pi T^3_3 = e^{-\alpha}\left[\beta''/2 - \alpha'\beta'/4 + \beta'^2/4 + (\beta' - \alpha')/2r\right]$

(2.5) $8\pi T^4_4 - e^{-\alpha}(\alpha'/r - 1/r^2) + 1/r^2$

where a prime denotes differentiation with respect to r. Thoughout the investigation we set velocity of light c and gravitational constant k to be unity. The energy momentum tensor T^i_j is given by

(2.6) $T^i_j = (\rho + p)u^i u_j - p\delta^i_j$

The above field equations for anisotropic fluid sphere provide us

(2.7) $e^{-\alpha}(1/r^2 - \alpha'/r) - 1/r^2 = -8\pi\rho$

(2.8) $1/r^2 - e^{-\alpha}(1/r^2 + \alpha'/r) = -8\pi p_r$

(2.9) $e^{-\alpha}[1/4\beta'\alpha' - 1/4\beta'^2 - 1/2\beta'' - 1/2(\beta' - \alpha')/r)] = -8\pi p_\perp$

where ρ is is energy density and p_1 and $p_\perp$ are the radial and tangential "pressure" respectively.

For the spherically symmetric metric (2.1) non-vanishing components of the Wely tensor are

(2.10) $C_{1212} = 1/12\, r\beta' + 1/12\, r\alpha' - 1/6 e^{\alpha} + 1/6 - 1/24 r^2\alpha'\beta'$

$$+1/24\, r^2\beta'^2 + 1/12 r^2\beta''$$

$$C_{1313} = \sin^2\theta\, C_{1212},\ C_{1010} = 2\, e^{\beta}/r^2\ C_{1212}$$

$$C_{2323} = -2\sin^2\theta e^{-\alpha} r^2 C_{1212},\ C_{2020} = e^{\beta-\alpha}\ C_{1212}$$

$$C_{3030} = -\sin^2\theta e^{\beta-\alpha}\ C_{1212}$$

We assume that the space time is conformally flat for which vanishing of Weyl tensor [22] gives

(2.10) $e^{\alpha}/r^2 - \frac{1}{r^2} - \beta'^2/4 + \beta'\alpha'/4 - \beta''/2 - \frac{1}{2r}(\alpha' - \beta') = 0$

Now we use the transformations

(2.11) $e^{-\alpha} = \tau$

(2.13) $\beta = 2\log y$

(2.13) $r^2 = z$

So that equations (2.8), (2.9) and (2.10) may be combined to give.

(2.14) $z\tau, z + 1 - \tau - 4\pi z(p_1 - p_{\perp}) = 0$

(2.15) $(4\tau z^2)y, z\ z + (2z^2\tau, z - \tau + 1)y = 0$

where the subscript z following a comma denotes differentiation with respect to z. Integrations of equations (2.14) and (2.15) give

(2.16) $\tau = e^{-\alpha} = 1 + \lambda r^2 + 8\pi r^2 \int_0^r [(p_r - p_{\perp})/r] dr$

(2.17) $y^2 = e^{\beta} = r^2[C\ e^{\zeta(r)} + De^{-\zeta(r)}]^2$

$$= r^2[C\ \exp(\zeta(r)) + D\exp(-\zeta(r))]$$

where λ, C and D are the integration constants and

(2.18) $\zeta(r) = \int \frac{e^{\alpha/2}}{r} dr$

The constants λ, C and D can be fixed by matching the metric functions (2.16) and (2.17) to the exterior Schwarzschild solution for a mass M and readius to given by

(2.19) $C = \frac{e\zeta^{\frac{(r_0)}{2}}}{2r_0}\left(\frac{3M}{r_0} - 1 + \left(\frac{1-2M}{r\varnothing}\right)^{3/2}\right)$

(2.20) $$D = \frac{e\zeta^{(r_0)/2}}{2r_0}\left(\frac{1-3M}{r_0} + \frac{(1-2M)^{3/2}}{r_0}\right)$$

(2.21) $$e^{-\alpha(r_0)} = \frac{(1-2M)}{r_0}$$

3. Solution of the Field Equations

We see that equations (2.7) – (2.9) and (2.16), (2.18) are actually three equations in four unknowns $\rho, p_r, p_\perp$ and $\zeta(r)$. Thus the system is indeterminate. To make the system determinate, we require one more relation or condition. For this we choose any of these unknowns as a function of r or by specifying an equation of state for the stresses.

Cases 1.

Here we choose the energy density as

(3.1) $$8\pi\rho = \frac{3\pi}{(1+\pi r^2)}\left(\frac{1}{1+\lambda r^2} + \frac{1}{2}\right)$$

where λ is a constant to be fixed up by boundary Conditions.

This distribution has been already considered by Durgapal and Banerjee [9] for the perfect fluid solution. Now using (3.1) into (2.16) we get

(3.2) $$e^{\alpha} = \frac{2(1+\mu x)}{2-\mu x}$$

where

(3.3) $$x = \frac{r^2}{r_0^2}$$

(3.4) $$\mu = \frac{4M/r_0}{3-4M/r_0}$$

and M and r_o are the mass and radius of the sphere.

Also the function $\zeta(r)$ is given by

(3.5) $e^{-\zeta}r_0^2 = \exp\left[\sqrt{2}\sin^{-1}\left(\frac{1-2\mu x}{3}\right)\right]$

$$\frac{X[4+\mu x+2\sqrt{2}\eta]}{x}$$

with

(3.6) $\eta^2 = 2+\mu x-\mu^2 x^2$

Putting this into equation (2.17) and using C and D from (2.19) and (2.20) we can find e^{β}. The density, radial pressure and tangential pressure are obtained as

(3.7) $8\pi\rho r_0^2 = \frac{\mu}{1+\mu x}\left(\frac{1}{1+\mu x}+\frac{1}{2}\right)$

(3.8) $8\pi p r_0^2 = \frac{Cx}{D}e^{-\zeta}r_0^2\frac{\left[4+4x(9-5h+2\sqrt{2}t)-5\mu x-2\sqrt{x})\right]}{2xh\left(\mu x+\frac{CX}{D}e^{-\zeta}r_0^2\right)}$

(3.9) $8\pi p_{\perp} r_0^2 = \frac{3\mu^2 x}{h^2}+\frac{Cx}{D}e^{-\zeta}r_0^2\frac{[9+\zeta h-2\sqrt{2t}+\mu x(9-5h-2\sqrt{2t}]}{2xh\left(\mu x+\frac{Cx}{B}e^{-\zeta}r_0^2\right)}$

Case II : Here we choose

(3.10) $\tau = \tau^{-\alpha} = \frac{(1+k_1 r^m)^n}{(1-k_2 r^m)^n}$

where k_1 and k_2 are constants. Then we can find e^{β}, ρ, p_r and $p_{\perp}$ from the field equations and using (2.15)-(2.21). Hower, for

mathematical simplicity we particularize m = n = 2 and k_i = A, k_j 3A

In this case we find the solution (by suitable adjustment of constants) as

(3.11) $e^{\alpha/2} = \frac{1+3\upsilon x}{1-\upsilon x}$

(3.12) $$e^{\beta/2} = \frac{(1-3\upsilon)(1+\upsilon)(1-\upsilon x)^4 + 4\upsilon x(1-\upsilon)^4}{[(1+3\upsilon)(1-\upsilon)(1-\upsilon x)]^2}$$

(3.13) $$8\pi\rho r_0^2 = \frac{8\upsilon(3+2\upsilon x+3\upsilon^2 x^2)}{(1+3\upsilon x)^2}$$

(3.14) $$8\pi P_r r_0^2$$

$$= \frac{16\upsilon[\{(1-\upsilon)^4/(1-3\upsilon)(1+\upsilon)-1)+[(1-2\upsilon x-3\upsilon^2 x^2)-(1-\upsilon x)^4]}{[4\upsilon\{(1-\upsilon)^4/(1-3\upsilon)(1+\upsilon)+1\}x(1-\upsilon x)^2](1+3\upsilon x)^2}$$

(3.15) $$-8\pi p_\perp r_0^2 = \frac{16\upsilon^2 x(5+3\upsilon x)}{(1-3\upsilon x)^2} + 8\pi p_r r_0^2$$

where $x = \dfrac{r^2}{r_0^2}$

$$\upsilon = \frac{1-\bar{M}}{1+3\bar{M}}$$

$$\bar{M} = \frac{1-2M}{\sqrt{r_0}}$$

Case III : If we choose the equation state

$$p_r = p_\perp$$

Then in this case the well known Schwarzschild interior solution is obtained.

4. Discussion

Here we have obtained two analytical solutions of Einstein's field equations for static fluid spheres with anisotropic pressure which are exact. Both the solutions I and II are free from singularities and densities of these fluid sphere drop continuously from their maximum values at the centre to values which are positive at the boundary. Again these solutions have reasonable equation of state for masses less than about 0.42 and 0.435 times the radius of fluid sphere (in geometric units) respectively. These solutions may be used in describing ultracompact objects [13]. From equations (3.9) and (3.5) we see that for the two models

$p_r = p_\perp$ at the centre when as $p_i = p_r$ for r > 0. Further it is easy to prove that in the low mass limit, m << r_0 the solutions in case I and Ii and the constant density solution of Schwarzschild have (to the first order approximation in M/r_0) the same common limit given by

$$(4.1)\quad e^{-\alpha} = 1 - \left(\frac{2M}{r_0}\right) x,$$

$$(4.2)\quad e^{\beta} = 1 - \left(\frac{M}{r_0}\right)(3 - x)$$

Where as in case I and II

$$x = \frac{r^2}{r_0^2}$$

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