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Stability of Liberation Points for Dumb-bell Artificial Satellites in Zero Eccentric Orbit by Lyapunov's Method

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Abstract

This work aims to investigate stability of Liberation points for non-linear motion of dumbbell satellites. These satellites are inter - connected by an inelastic tether. The tether is also assumed as non-conducting, light and flexible. The system is affected by various influencing forces. The perturbative forces which are occurred by the earth itself named as magnetic force, oblateness and shadow. The rest two perturbations are due to air drag and radiation pressure of the sun. We have investigated Liberation point of the system moving in orbit having eccentricity value zero. To examine whether this Liberation point is stable or unstable, Lyapunov's method has been applied. We investigated that obtained Liberation point is unstable according to Lyapunov's method.

Key-words: Dumb-bell, Equilibrium position, Perturbations, Lyapunov's method, Zero eccentric orbit,

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1. Introduction

In general, the problems of space physics and dynamics are described by a system of ordinary differential equations of second order. We find out the first integral of the differential equations of motion. Unfortunately, in almost all the cases their does not exist enough independent first integrals of motion. The second method of approach is to find out series solution of the problem by some well-known methods of perturbations theories developed by several researchers. As a matter of fact, the series so developed exhibit very slow convergence and hence the result so obtained does not remain valid for long period of time. However, the work often describes very accurate result for a limited time interval and hence the work is frequently used for determination of the orbits of artificial satellite.

Researches in the branch of non-linear dynamics attached through a tether was initiated by a team of Russian scientists[2-3]. Later, the problem was investigated by Singh and Demin[4] in two-dimensional case and generalized by Singh[5] in three-dimension as well. Further, the analysis of a tether attached system was studied by so many workers[6-7]. Kurpa et-al[8] handle the problem of the concept of space flight. Khan and Goel[9] studied about the chaotic motion of dumb- bell satellite. Several authors[10-13] could study the non-linear dynamics of artificial satellites attached through a tether under the combined effect of different perturbating forces. Recently, several researchers[15-17] are utilizing technology of coupled satellites in various space applications.

This work is more general as compared to the previous works in the sense that we are dealing with five different perturbative forces. These forces will act on the system simultaneously. The previous workers could handle the problem by considering only one and by a combination of any two or three or four perturbations. Therefore, a clear view of dynamics of the satellites was not observed. Here both the satellites are treated as charged particles. Forces of attraction working between both satellites are neglected because distances among the celestial bodies and the satellites are very large.

2. Mathematical Treatment:

Second order non-linear differential equations for the system in rotating frame for two satellites of masses m_1 and m_2 with charges Q_1 and Q_2 is written as Kumar and Kumar[14]

$$X'' - 2Y' - 3X = \beta X - A\cos i + \left(\frac{B_1}{m_1} - \frac{B_2}{m_2}\right).\frac{\sin\theta_2 \cos \in \cos\alpha}{\pi} + \frac{12\mu K_2}{R^5}X$$

As well as

$$Y'' + 2X' = \beta Y + \left(\frac{B_1}{m_1} - \frac{B_2}{m_2}\right).\frac{\sin\theta_2 \cos\in\sin\alpha}{\pi} - \frac{3\mu K_2}{R^5}Y - f$$

(1)

Condition of constraint is as follows

$$X^2 + Y^2 \leq 1 \qquad (2)$$

Here,

$$A = \left(\frac{m_1}{m_1+m_2}\right)\left(\frac{Q_1}{m_1} - \frac{Q_2}{m_2}\right)\frac{\mu_E}{\sqrt{\mu\rho}}, \beta = \frac{p^3\lambda}{\mu}\left(\frac{m_1+m_2}{m_1 m_2}\right),$$

$$\rho = \frac{1}{(1+e\cos\nu)}, f = \frac{a_1 p^3}{\sqrt{\mu p}},$$

$$a_1 = \rho_a \dot{R}(c_2 - c_1)\left(\frac{m_1}{m_1+m_2}\right)$$

$$k_2 = \frac{\in R_e^2}{3}, \in = \alpha_R - \frac{m}{2}, m = \frac{\Omega^2 R_e}{g_e} \qquad (3)$$

Where

α_R denotes the earth oblateness, Ω refers to angular velocity, R_e represents radius of the earth. B_1 and B_2 indicate direct radiation pressure of the sun. The focal parameter is denoted by p. μ_E= earth's magnetic moment, e = eccentricity of the orbit, $\in$ = inclination of the oscillatory plane and v = true anomaly. λ shows undetermined Lagrange's multiplier. α is the ray's inclination. R is the modulus of position vector of C.M. c_1 and c_2 refers to Ballistic coefficients. ρ_a indicates average density of atmosphere. i denotes orbital gradient. γ = shadow function which have values 0 and 1. Here (') represents differentiation w.r.t. v.

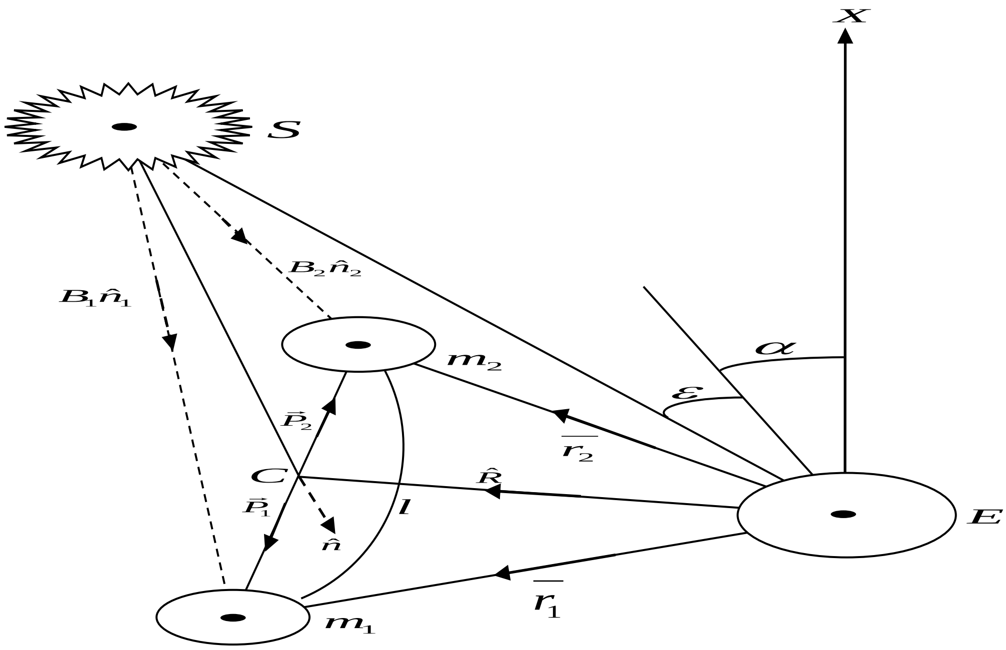

Figure. 1: Diagrammatical representation of Dumb-bell Artificial Satellites under several influences

We shall have Jacobian integral of the system corresponding to above set of non-linear differential equations of motion (1) as

$$X'^2+Y'^2-3X^2=\beta-2AX\cos i+\frac{2}{\pi}\left(\frac{B_1}{m_1}-\frac{B_2}{m_2}\right)\left(X\cos\alpha+Y\sin\alpha\right)\sin\theta_2\cos\in$$
$$+\frac{3\mu K_2}{R^5}\left(4X^2-Y^2\right)-2fY+h \tag{4}$$

In equation (4) h is termed as Jacobian constant.

3. Calculation of Liberation Points:

To find out the Liberation point, we substitute

$$X = X_0 \quad \text{and} \quad Y = Y_0 \tag{5}$$

Here, X_0 and Y_0 are constant and these will lead to equilibrium positions.

Next,

$$X' = X'_0 = 0 \; ; \qquad X'' = X''_0 = 0$$

$$Y' = Y'_0 = 0 \; ; \qquad Y'' = Y''_0 = 0 \tag{6}$$

Substituting (6) in (1), the result comes to

$$(3+\beta) X_0 = A\cos i - \left(\frac{B_1}{m_1} - \frac{B_2}{m_2}\right) . \frac{\sin\theta_2 \cos\in \cos\alpha}{\pi} - \frac{12\mu K_2}{R^5} X_0$$

and

$$\beta Y_0 = f - \left(\frac{B_1}{m_1} - \frac{B_2}{m_2}\right) . \frac{\sin\theta_2 \cos\in \sin\alpha}{\pi} + \frac{3\mu K_2}{R^5} Y_0 \tag{7}$$

To obtain the solutions of equations (7) is very difficult. That is why, we confine ourselves to certain limitations. As $\left(\frac{B_1}{m_1} - \frac{B_2}{m_2}\right)$ or θ_2 cannot be zero, we put $\in = 0 \;\&\; \alpha = 0$. If not so then complication exists for further simplification of the problem.

Therefore, equations (7) will be written as

$$(3+\beta)X_0 = A\cos i - \left(\frac{B_1}{m_1} - \frac{B_2}{m_2}\right).\frac{\sin\theta_2}{\pi} - \frac{12\mu K_2}{R^5}X_0$$

and

$$\beta Y_0 = f + \frac{3\mu K_2}{R^5}Y_0$$

(8)

Now, equations (8) will give the required equilibrium position as:

$$[X_0, Y_0] = \left[\frac{A\cos i - \left(\frac{B_1}{m_1} - \frac{B_2}{m_2}\right).\frac{\sin\theta_2}{\pi}}{\left(3+\beta+\frac{12\mu K_2}{R^5}\right)}, \frac{f}{\left(\beta - \frac{3\mu K_2}{R^5}\right)}\right]$$

(9)

4. Stability of Liberation Point:

Liapunov's theorem[1] is applied to examine stability of the obtained equilibrium position (9). We assume δ_1 and δ_2 as small variations at the Liberation point.

Therefore, we write

$$X = X_0 + \delta_1 \qquad Y = Y_0 + \delta_2$$

$$\therefore X' = \delta_1' \qquad Y' = \delta_2'$$

$$\therefore X'' = \delta_1'' \qquad Y'' = \delta_2'' \tag{10}$$

Using (10) in the equations (1), a set of variational equations comes to the form

$$\delta''_1 - 2\delta'_2 - (X_0 + \delta_1)\left(3+\beta+\frac{12\mu k_2}{R^5}\right) = \left(\frac{B_1}{m_1} - \frac{B_2}{m_2}\right).\frac{\sin\theta_2}{\pi} - A\cos i$$

and

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$$\delta_2'' + 2\delta_1' = (Y_0 + \delta_2)(\beta - \frac{3\mu k_2}{R^5}) - f \tag{11}$$

Where $\in = 0$ and $\alpha = 0$

As the initial equations (1) describes Jacobian integral, the obtained variational equations of motion (11) will also exhibit Jacobian integral.

On solving the first and second equations of (11) by multiplying them $2\delta_1'$ and $2\delta_2'$ respectively. Consequently, adding and integrating obtained equation, we get Jacobean Integral as

$$\delta_1'^2 + \delta_2'^2 + \delta_1^2\left(-3 - \beta - \frac{12\mu k_2}{R^5}\right) + \delta_2^2\left(-\beta + \frac{3\mu k_2}{R^5}\right)$$

$$+\delta_1\left[2\left\{A\cos i - \left(3 + \beta + \frac{12\mu k_2}{R^5}\right)X_0 - \left(\frac{B_1}{m_1} - \frac{B_2}{m_2}\right).\frac{\mathrm{Sin}\,\theta_2}{\pi}\right\}\right] + \delta_2\left[2\left\{f - \beta Y_0 + \frac{3\mu k_2}{R^5}Y_0\right\}\right] = h_1 \tag{12}$$

Where h_1 is constant of integration.

For testing the stability by Lyapunov's theorem, Jacobean integral is considered as

Lyapunov function $L(\delta_1', \delta_2', \delta_1, \delta_2)$

Hence,

$$L(\delta_1', \delta_2', \delta_1, \delta_2) = \delta_1'^2 + \delta_2'^2 + \delta_1^2\left(-3 - \beta - \frac{12\mu k_2}{R^5}\right) + \delta_2^2\left(-\beta + \frac{3\mu k_2}{R^5}\right)$$

$$+\delta_1\left[2\left\{A\cos i - \left(3 + \beta + \frac{12\mu k_2}{R^5}\right)X_0 - \left(\frac{B_1}{m_1} - \frac{B_2}{m_2}\right).\frac{\mathrm{Sin}\,\theta_2}{\pi}\right\}\right] + \delta_2\left[2\left\{f - \beta Y_0 + \frac{3\mu k_2}{R^5}Y_0\right\}\right] \tag{13}$$

5. Results and discussion:

According to Liapunov's theorem, "L is taken as Lyapunov's function''. The differentiation of Lyapunov's function along the path of the system vanish identically. Thus, for stability of Liberation point in Lyapunov's method, it is required that Lyapunov function must has positive value. For that reason, in equation (13) terms of first order variables must be zero whereas the terms of second order variables must fulfil Sylvester's conditions. Therefore, the sufficient conditions for the stability are as follows

(i) $\left\{A\cos i-\left(3+\beta+\frac{12\mu k_2}{R^5}\right)X_0-\left(\frac{B_1}{m_1}-\frac{B_2}{m_2}\right).\frac{\text{Sin}\,\theta_2}{\pi}\right\}=0$

(ii) $\left\{f-\beta Y_0+\frac{3\mu k_2}{R^5}Y_0\right\}=0$

(iii) $-\left(3+\beta+\frac{12\mu k_2}{R^5}\right)>0$ and

(iv) $(-\beta+\frac{3\mu k_2}{R^5})>0$

(14)

6. Conclusion:

Investigating the conditions (14) separately, we find that all the four restrictions of stable Liberation point (X_0, Y_0) are not identically satisfied simultaneously. Therefore, the equilibrium position is unstable according to Lyapunov's Method.

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