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A STUDY OF *K*-FRAMES IN HILBERT AND BANACH SPACES AND ITS PROPERTIES

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Abstract: It is known that frames on a independent Hilbert space H and constructive invertible delimited operators have a close relationship. This study demonstrates that K-frames and quotient operators have a relationship on behalf of a bounded machinist K on H. Găvruţa created K-frames in Hilbert to investigate atomic classification in relation in the direction of a delimited linear operator. The further discussed in this study because of the numerous distinctions amid K-frames as well as ordinary frames that she highlighted. We also provide a number of ways to build K-frames. Operator-theoretic theorems on the quotient of bounded operators have been used to prove conclusions on K-frames. As a final point, we address steadiness of a more generic K-frame perturbation.

Keywords: Banach Spaces, Hilbert Spaces, *K*-FRAMES, finite rank

1. Introduction

Frames are extensions of orthnormal center in Hilbert spaces. The chief characteristic of frames that adds value is their redundancy. In many real-world scenarios, employing frames instead than basis expansions to express signals is preferable. Because of its numerous properties, frames are useful in a wide range of scientific, technical, and mathematical applications. Geophysics, quantum work out, wireless feeler set-up, and many more areas are among the numerous domains in which frames are especially helpful. Frames are generalized from orthonormal bases. Linear independence condition in favor of a basis agrees to each vector to be exceptionally represent as a linear combination, hence severely restricting practical challenges. Frames consent to any constituent in space to subsist described as a linear combination of components in frame. The operator K was invented and investigated by Holub (1994). K-frames and regular frames differ greatly in many key aspects. For instance, we are aware that a key analogous feature of ordinary frames be that their similar synthesis machinist is delimited and subjective. On the other hand, the *K* range must be contained in the ranges of the synthesis operators, and the corresponding synthesis operators must be constrained for *K*-frames. Găvruţa developed a generalization of earlier frames in literature: a frame for operators in Hilbert spaces. Its relationship to operators is the rationale behind its selection and study in thesis. We construct frame succession and investigate a group of pupil of operator associated with a particular Bessel sequence, which transforms it into a frame on behalf of all operators in class.

2. Literature of Review

Aldroubi (1995) we present two ways to create frames in a space H. Enclosed operators on H are used in the first approach. The other approach creates frames of H by applying limited linear operators on l2. We provide some examples from wavelet and multiresolution theory to demonstrate the results. **Alizadeh et al. (2021)**The purpose of this paper is to define duality c-K-g-fusion frames and characterise types of duals, define notion c-K-g fusion edge which be generalisations K-g-fusion edge establish a quantity of novel outcome on c-K-g-fusion casing in Hilbert, and discuss perturbation c-K-g-fusion frames. An example from the affine group is used to highlight and demonstrate the theory's generalisations. **Colombo et al. (2014)** We present and examine a functional calculus for bi complex linear bounded operators in this paper. The study's foundation is the two non real idempotent's decomposition of linear operators and bi complex numbers. Consequently, we present an alternative spectrum, referred to as the reduced spectrum, which is bounded and proves to be the appropriate instrument for building the bi complex holomorphic functional calculus. **Deepshikha et al. (2017)** This study investigates the operator-based characterisation of weaving Δ-g-frames is provided. It is demonstrated that Δ-g-woven frames yield regular knit Θ-frames if the edge boundaries of the frames connected to atomic spaces are positively limited, and vice versa. **Ding (2011)**The concept of generalised continuous Kframe in a Hilbert space is introduced in this note. There are examples provided to show that generalised continuous K-frames exist. Lastly, we investigate K-frame perturbation and derive requirements for the stability of generalised continuous K-frames, keeping in mind the significance of perturbation theory in numerous fields of applied mathematics. **Goswami and Pathak (2019)** With some illustrated instances, we characterize and investigate the near-exact Λ-Banach enclose on behalf of machinist spaces in this paper. There is now a sufficient requirement for a ΛBanach frame to be almost accurate.

Additionally, we present a way to build κ-Banach frames related to O-frames. Furthermore, a few findings of O-frame characterisations have been provided. **Hong and Li (2023)** The thought of transmit fusion edge and Banach transmit fusion edge in spaces are presented in this article, along with some of the appealing characteristics of relay fusion frames in this broader context. Additionally, Banach relay fusion frames and relay fusion frames' stability issue will be resolved. **Khosravi and Musazadeh (2008).** generalize a quantity of of notorious conclusions in frame speculation to fusion edge as well as g-frames, which were previously thought of as generalizations of frames in Hilbert spaces. By examining g-frames for their constituent parts, we derive new g-frames. Additionally, we derive some findings regarding the stability of g-frames under minor perturbations, excess of g-frames, and alternate dual g-frames. **Kaushik et al. (2014)** presented and examined the Banach space reconstruction property. Enough requirements are obtained in this study on behalf of the renovation belongings to be in Banach spaces. The scheme of Besselian sort renovation goods in Banach spaces is set up and its submission to Banach is established, motivated by a publication of Holub. **Krishna (2023)** Group-frames, or frames created by unitary representations of groups, are exclusively examined for Hilbert spaces in the literature. Using the representation's double commutant, a sizable set of functional-vector pairs that produce group-frames for Banach spaces is found. We derive in functional form the Ron-Shen duality, Wexler-Raz criterion, Moyal formula, and the fundamental identity of Gabor analysis. **Singh et al. (2019).** The maximum number of elements that can be eliminated from a frame while still leaving a set that serves as a frame for the underlying space is known as the excess of a frame. A description of retro Banach frames in Banach spaces with finite excess is presented. We derive a sufficient condition for the existence of an infinitely excess retro Banach frame.

3. Objective

- To find the Study of *K*-Frames in Hilbert and Banach Spaces and its Properties

4. Results and Discussions

The non-surjective conforming synthesis operator, the non-isomorphic frame operator, and the non-interchangeable alternate dual reconstruction pair are some of the features of regular frames that may not be applicable to *K*-frames. Frames are generalized from orthonormal basis. Linear independence properties of a base, which allow all vectors to stand exceptionally

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described by means of a linear mixture, significantly restrict practical challenges. One element in the planetary can be depicted as a lined combination of the fundamentals in the border thanks to surrounds, even though line independence among frame basics be not obligatory. They present trajectories in a stable, basis-like representation of Hilbert space. From a theoretical and practical perspective, the frame operator is a fundamental idea in frame theory. A frame operator is defined as a bounded confident invertible operative. However, building a frame that corresponds to a certain circumscribed confident invertible machinist scheduled a Hilbert has important realistic implications. Given a bounded constructive operator A scheduled H, here be a delimited encouraging operator B taking place H s.t. in situation of intangible frame. B be the constructive square root of A, that's why $A = B^2$. Frame machinists of orthonormal origin $\{e_i\}_{i=1}^{\infty}$ as well as Riesz basis exist I as well as BIB separately. Later A be frame machinist of frame $\{Be_i\}_{i=1}^{\infty}$.

Definition 1: A frame $\{f_k\}$ on behalf of $\mathcal{H}$ is assumed to exist

(i) Besselian if when on earth $\sum_{k=1}^{\infty} a_k f_k$ converges, $\{a_k\} \in \ell^2$.

(ii) a by Riesz basis stipulation there be a finite set σ on behalf of which $\{f_k\}_{k \notin \sigma}$ is a Riesz foundation on behalf of $\mathcal{H}$.

As stated in solitary of fundamental outcome (1994), "A frame on behalf of Hilbert space be Besselian if it is a near-Riesz basis," Motivated via a technique that describes a prearranged frame in a Hilbert as descriptions of a whole orthonormal bargain under a measure map, we also present the Besselian category renovation feature in Banach spaces.

Definition 2: For $\mathcal{X}$, a reconstruction system $(\{f_k\}, \{f_k^*\})$ is definite as

(i) $\mathcal{X}$-Besselian if $\sum_{k=1}^{\infty} |f_k^*(f)|^2 < \infty$, $\forall f \in \mathcal{X}$……(1)

(ii) $\mathcal{X}^*$-Besselian I f $\sum_{k=1}^{\infty} |f^*(f_k)|^2 < \infty$, $\forall$ l $f^* \in \mathcal{X}^*$…..(2)

(iii) $\mathcal{X}^{**}$-Besselian if $\sum_{k=1}^{\infty} |\Phi(f_k^*)|^2 < \infty$, $\forall\ \Phi \in \mathcal{X}^{**}$…..(3)

Example 1: Let $\mathcal{X} = L^1(\Omega)$, Ω is set of constructive integers through including measure. Regard as a structure $\{f_k\} \subset \mathcal{X}$ specified by $f_1 = \chi_1, f_k = \chi_{k-1}$, where $\chi_k = \{0,0,0, \dots, \underbrace{1}_{k^{th}-\text{place}}, 0,0, \dots\}$. Explain $\{f_k^*\} \subset \mathcal{X}^*$ by $f_1^*(f) = 0, f_k^*(f) = \xi_{k-1}, f = \{\xi_j\} \in \mathcal{X}$.

It can therefore be confirmed that although $(\{f_k\},\{f_k^*\})$ is not $\mathcal{X}^*$--Besselian, it is $\mathcal{X}^*$-Besselian. Consequently, we infer as of this that

$$\mathcal{X}\text{ - Besselian } \not\Rightarrow \mathcal{X}^*\text{- Besselian } \not\Rightarrow \mathcal{X}^{**}\text{ - Besselian. ….(4)}$$

Proposition 1: If $(\{f_k\},\{f_k^*\})$ have rebuilding property on behalf of $\mathcal{X}$ in addition to $\widehat{\Theta} \in B(\mathcal{X},\ell^2)$ be s.t. $\widehat{\Theta}(f_k) = e_k$, $\forall, k \in \mathbb{N}$, before $(\{f_k\},\{f_k^*\})$ exists $\mathcal{X}$-Besselian, where $\{e_k\}$ is categorization of unit vectors in ℓ^2.

Proof: Subsequently $f = \sum_{k=1}^{\infty} f_{\ k}^{\ *}(f)\, f_k$, $f \in \mathcal{X}$,

$$\begin{aligned} \infty &> \|\widehat{\Theta}\,(f)\|^{\,2} \\ &= \|\textstyle\sum_{k=1}^{\infty}\, f_{\ k}^{\ *}(f)\, e_k\|^2 \qquad \text{….(5)} \\ &= \textstyle\sum_{k=1}^{\infty}\, |f_{\ k}^{\ *}(f)|^2, \text{ for all } f \in \mathcal{X} \end{aligned}$$

Therefore $(\{f_k\},\{f_k^*\})$ is $\mathcal{X}$-Besselian.

Proposition 2: Assume that $(\{f_k\},\{f_k^*\})$ have renovation property on behalf of $\mathcal{X}$. If there exists an operative $\widehat{\boldsymbol{\Theta}} \in B(\ell^2,\mathcal{X})$ s.t. $\widehat{\Theta}(e_k) = f_k$, $\forall\, k \in \mathbb{N}$, $(\{f_k^*\},\{f_k\})$ be $\mathcal{X}^*$ Besselian

Proof: We analyze

$$\begin{aligned} &\|\textstyle\sum_{k=1}^{n}\, f^*(f_k) f_k\| \\ &= \left\|\textstyle\sum_{k=1}^{n}\, f^*(f_k)\widehat{\boldsymbol{\Theta}}(e_k)\right\| \text{ ….(6)} \\ &\le \|\widehat{\boldsymbol{\Theta}}\|\|\textstyle\sum_{k=1}^{n}\, f^*(f_k) e_k\| \end{aligned}$$

At present for all $f^* \in \mathcal{X}^*$, by overriding (6), we have need of

$$\begin{aligned} &\sum_{k=1}^{n} |f^*(f_{\,k})|^{\,2} \\ &= f^*\left(\sum_{k=1}^{n} \overline{f^*\,(f_k)} f_k\right) \qquad \text{…. (7)} \\ &\le \|f^{\,*}\|\|\widehat{\Theta}\| \sqrt{\sum_{k=1}^{n} |f^*\,(f_k)|^2}, \forall n \in \mathbb{N} \end{aligned}$$

Consequently, $\sqrt{\sum_{k=1}^{n} |f^*\,(f_{\,k})|^{\,2}} \le \|\,f^*\|\|\widehat{\Theta}\|$, $n \in \mathbb{N}$ and $\forall f^{\,*} \in \mathcal{X}^{\,*}$. So, $\sum_{k=1}^{\infty} |f^*\,(f_{\,k})|^{\,2} < \infty$. Later structure $(\{f_{\,k}\},\{f_{\,k}^{\,*}\})$ is $\mathcal{X}^*$-Besselian. Assume X has

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reconstruction possessions on behalf of $(\{f_k\},\{f_k^*\})$, and $\{g_k\} \subset \mathcal{X}$ be, in some sense, near $\{f_k\}$. Thus, $(\{g_k^*\},\{\pi(g_k)\})$ do not function as a reconstruction possessions on behalf of $\mathcal{X}^*$ in general. It should be noted that in the aforementioned argument, selecting an arbitrary $\{\psi_k\} \subset \mathcal{X}^{**}$ in place of $(\{\pi(g_k)\})$ is equally admissible. In these circumstances, every element of $\mathcal{X}^*$ can be retrieved by a bounded linear operator (connected to $(\{\pi(g_k)\})$. As stated in solitary of fundamental outcome (1994), "A frame on behalf of Hilbert space be Besselian if it is a near-Riesz basis," Motivated via a technique that describes a prearranged frame in a Hilbert as descriptions of a whole orthonormal bargain under a measure map, we also present the Besselian category renovation feature in Banach spaces. Sufficient conditions aimed at the life of a renovation operator S s.t. are given by the following theorem. For $\mathcal{X}^*$,$(\{\pi(g_k)\},\mathcal{S})$ is a Banach frame.

Theorem 1: consent to $\{g_k\} \subset \mathcal{X}$ be s.t. on behalf of every $f \in \mathcal{X}$. Assume that $(\{f_k\},\{f_k^*\})$ has reconstruction possessions on behalf of $\mathcal{X}$ which be $\mathcal{X}$ -Besselian.

$$\Delta \times \sqrt{\sum_{k=1}^{\infty} |f_k^*(f)|^2} < \delta\|f\| \ \ldots(8)$$

$$\Delta = \sup_{\substack{\phi^* \in \mathcal{X}^* \\ \|\phi^*\| \leq 1}} \sqrt{\sum_{k=1}^{n} |\phi^*(f_k - g_k)|^2}, n \in \mathbb{N} \ldots(9)$$

Next, a reconstruction operator $\mathcal{S}$ is known to exist s.t.$(\{\pi(g_k)\},\mathcal{S})$ is a Banach frame on behalf of $\mathcal{X}^*$.

Proof: on behalf of apiece n indicate $\psi_n^* \in \mathcal{X}^*$ thru $\|\psi_n^*\| = 1$ s.t.

$$\psi_n^*\left(\sum_{k=1}^{n} f_k^*(f)(f_k - g_k)\right) = \left\|\sum_{k=1}^{n} f_k^*(f)(f_k - g_k)\right\| \ldots(10)$$

Consequently, $\forall\, f \in \mathcal{X}$, we require

$$\begin{aligned} &\left\|\sum_{k=1}^{n} f_k^*(f)(f_k - g_k)\right\| \\ &\leq \sqrt{\sum_{k=1}^{\infty} |f_k^*(f)|^2} \times \sqrt{\sum_{k=1}^{n} |\psi_n^*(f_k - g_k)|^2} \ldots(11) \\ &\leq \sqrt{\sum_{k=1}^{\infty} |f_k^*(f)|^2} \times \Delta \end{aligned}$$

Take on that there stands no reconstruction operative $\mathcal{S}$ s.t. $(\{\pi(g_k)\},\mathcal{S})$ stays a Banach frame on behalf of $\mathcal{X}^*$ w.r.t an allied sequence space $\mathcal{Z}_d$ there exists $f_0 \in \mathcal{X}$ s.t. $\|f_0\| = 1$ in addition to $\text{dist}(f_0,[g_k]) > \epsilon (0 \leq \epsilon < 1)$. Consequently, Hahn-Banach Theorem contributes a nonzero functional $\psi^* \equiv \psi^*_{f_0} \in \mathcal{X}^*$ s.t. $\psi^*(g_k) = 0, k \in \mathbb{N}; \psi^*(f_0) = 1;$ $\|\psi^*\| = \frac{1}{\text{dist}(f_0,[g_k])}$. By consuming (12), we require

$$\begin{aligned} 1 &= |\psi^*(f_0)| \\ &= \lim_{n\to\infty} |\psi^*(\textstyle\sum_{k=1}^{n} f_k^*(f_0) f_k)| \\ &= \lim_{n\to\infty} \left|\psi^*\left(\textstyle\sum_{k=1}^{n} f_k^*(f_0)(f_k - g_k)\right)\right| \quad \ldots.(13) \\ &\leq \lim_{n\to\infty} \|\psi^*\| \|\textstyle\sum_{k=1}^{n} f_k^*(f_0)(f_k - g_k)\| \end{aligned}$$

In specific for $\delta = \epsilon$, (13) gives $\text{dist}(f_0, [g_k]) \leq \epsilon$, a inconsistency. Assume that given $\mathcal{X}$, $(\{f_k\}, \{f_k^*\})$ possesses the reconstruction feature. Next, a reconstruction operator $\mathcal{S}$ s.t. can be located. A Banach frame for $\mathcal{X}$ is $(\{f_k^*\}, \mathcal{S})$, and it is related to the system $(\{f_k\}, \{f_k^*\})$. In the same way, we may identify a reconstruction operator linked to system $\{f_k\}$. Linear independence properties of a base, which allow all vectors to stand exceptionally described by means of a linear mixture, significantly restrict practical challenges. One element in the planetary can be depicted as a lined combination of the fundamentals in the border thanks to surrounds, even though line independence among frame basics be not obligatory. They present trajectories in a stable, basis-like representation of Hilbert space. From a theoretical and practical perspective, the frame operator is a fundamental idea in frame theory. A frame operator is defined as a bounded confident invertible operative. However, building a frame that corresponds to a certain circumscribed confident invertible machinist scheduled a Hilbert has important realistic implications. The question of whether we know how to locate Banach frames on behalf of a broad class of spaces connected to a certain reconstruction structure naturally arises. We present Banach Λ-frames in favor of the machinist spaces in this direction.

Definition 3: Consent to $\mathcal{Y}_d$ be a sequence space related to $\mathcal{Y}$, and $\mathcal{X}$ plus $\mathcal{Y}$ Banach spaces. In case of $\mathcal{B}(\mathcal{X},\mathcal{Y})$, structure $\{f_k\} \subset \mathcal{X}$ be referred to as a Banach Λ-frame if $0 < \mathrm{a}_0 \leq \mathrm{b}_0 < \infty$ s.t.

$$\mathrm{a}_0\|\Lambda\| \leq \|\{\Lambda(f_k)\}\|_{\mathcal{Y}_d} \leq \mathrm{b}_0\|\Lambda\|, \forall\, \Lambda \in \mathcal{B}\,(\mathcal{X},\mathcal{Y}) \ldots\ldots (14)$$

Proposition 3: Assume that $\{f_k^*\} \subset \mathcal{X}^* \setminus \{0\}$ has renovation property on behalf of $\mathcal{X}$ w.r.t $\{f_k\} \subset \mathcal{X}$ after that, $\{f_k\}$ stands a Banach Λ-frame on behalf of operator space $\mathcal{B}(\mathcal{X},\mathcal{V})$ per esteem to $\mathcal{V}_d$.

Proof: Let $\Lambda \in B(\mathcal{X},\mathcal{V})$ exist indiscriminate in support of each $n \in \mathbb{N}$, outline $\Lambda_n: \mathcal{X} \to \mathcal{V}$ via

$$\Lambda_n(f) = \sum_{k=1}^{n} f_k^*(f)\Lambda(f_k), f \in \mathcal{X} \ldots.(15)$$

Before

$$\lim_{n\to\infty} \Lambda_n(f) = \lim_{n\to\infty} \sum_{k=1}^{n} f_k^*(f)\Lambda(f_k) \ldots.(16)$$

$$= \Lambda(f)$$

So, $\sup_{1\le n<\infty} \|\Lambda_n(f)\| < \infty, \forall f \in \mathcal{X}$. Consequently, by proposition of Banach-Steinhaus, we require $\sup_{1\le n<\infty} \|\Lambda_n\| < \infty$, Fix $\Lambda \in \mathcal{B}(\mathcal{X},\mathcal{V})$. Before,

$$\begin{aligned}\|\Lambda\| &= \sup_{\substack{f\in\mathcal{X}\\ \|f\le 1\le 1}} \|\Lambda(f)\| \\ &= \sup_{\substack{f\in\mathcal{X}\\ \|f\|\le 1}} \|\Lambda(\textstyle\sum_{k=1}^{\infty} f_k^*(f) f_k)\| \\ &= \sup_{\substack{f\in\mathcal{X}\\ \|f\|\le 1}} \left\|\lim_{n\to\infty} \textstyle\sum_{k=1}^{n} f_k^*(f)\Lambda(f_k)\right\| \ldots.(17) \\ &\le \sup_{1\le n<\infty} \|\Lambda_n\|\end{aligned}$$

Likewise $\forall f \in \mathcal{X}$, we require

$$\|\sum_{k=1}^{n} f_k^*(f)\Lambda(f_k)\|$$

$$= \|\Lambda\|\|P_n(f)\| \ldots..(18)$$

$$\text{where } B = \sup_{1\le n<\infty} \|P_n\|$$

Thus, by consuming (18) we takesup $_{\substack{f\in X\\ \|f\|\le 1}} \|\sum_{k=1}^{n} f_k^*(f)\Lambda(f_k)\| \le B\|\Lambda\|$. This affords

$$\|\{\Lambda(f_k)\}\|_{\mathcal{V}_d} = \sup_{n} \sup_{\substack{f\in\mathcal{X}\\ \|f\|\le 1}} \|\sum_{k=1}^{n} f_k^*(f)\Lambda(f_k)\|_{\mathcal{V}} \ldots.(19)$$

By consuming (17) and (19) thru $A = 1$, we require

$$A\|\Lambda\| \le \|\{\Lambda(f_k\}\|\mathcal{V}_d \le B\|\Lambda\|, \forall \Lambda \in B(\mathcal{X},\mathcal{V}) \ldots.(20)$$

Therefore $\{\Lambda(f_k)\}$ be a Banach Λ-frame on behalf of operator space $\mathcal{B}(\mathcal{X},\mathcal{V})$ thru reverence to $\mathcal{V}_d$. This finishes the evidence.

4.1 $\mathcal{X}_d$-Atomic Systems and $\mathcal{X}_d$ - K-frames

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A sequence of essentials in $\mathcal{X}^*$ rather than basics in original space $\mathcal{X}$, was defined as a frame on behalf of a Banach $\mathcal{X}$. However, the creation of inner product sort influence in spaces is made possible by semi-inner products for Banach spaces. Dastourian and Janfada presented the idea of relatives of local bit in a space $\mathcal{X}$ w.r.t a BK-space $\mathcal{X}_d$ using a semi-inner product. This idea led to construction of a novel frame by way of regard to the machinist K, known as the $\mathcal{X}_d^*$- K-frame, and was comprehensive to a minuscule system on behalf of a machinist K∈B(X), dubbed the $\mathcal{X}_d^*$-atomic structure. In contrast to the conventional method of analyzing string in dual space $\mathcal{X}^*$, semi-inner products are used in analyse string in the innovative space $\mathcal{X}$ in order to examine a ancestors of $\mathcal{X}_d^*$-local atoms in addition to $\mathcal{X}_d^*$-atomic structure. Linear independence properties of a base, which allow all vectors to stand exceptionally described by means of a linear mixture, significantly restrict practical challenges. A frame operator is defined as a bounded confident invertible operative. However, building a frame that corresponds to a certain circumscribed confident invertible machinist scheduled a Hilbert has important realistic implications. At this time, $\mathcal{X}$ be taken to exist a space that is reflexive and separable. To fit string in dual space without need of semiinner products, the description of $\mathcal{X}_d^*$--atomic structure and $\mathcal{X}_d^*$--K-frames, respectively, have undergone appropriate modifications. Therefore, one may consider the idea of $\mathcal{X}_d$-K-frames on behalf of Banach to be a generalisation of $\mathcal{X}_d$-frames.

Definition 4: Consent to $\mathcal{X}$ stand a Banach space in addition to $\mathcal{X}_d$ exist a BK-space. Consent to $K \in \mathcal{B}(\mathcal{X}^*)$ & $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^*$. We declare that $\{g_i\}_{i=1}^{\infty}$ be an $\mathcal{X}_d$-atomic structure on behalf of $\mathcal{X}$ w.r.t K if subsequent declarations clasp:

1. $\sum_{i=1}^{\infty} d_i\, g_i$ converges in $\mathcal{X}^*$ $\forall d = \{d_i\}_{i=1}^{\infty}$ in $\mathcal{X}_d^*$ & $\in \mu > 0$ s.t. $\|\sum_{i=1}^{\infty} d_i\, g_i\|_{\mathcal{X}^*} \leq \mu\, \|d\|_{\mathcal{X}_d^*}$;…..(21)
2. If $c > 0$ s.t. $\forall g \in \mathcal{X}^* \in a_g = \{a_i\}_{i=1}^{\infty} \in \mathcal{X}_d$ s.t. $\|a_g\|_{\mathcal{X}_d} \leq c\, \|g\|_{\mathcal{X}^*}$ in addition to $K\, g = \sum_{i=1}^{\infty} a_i g_i$….(22)

When $\mathcal{X}_d$ be reflexive, state (21) in in fact articulates that $\{g_i\}_{i=1}^{\infty}$ be $\mathcal{X}_d$-Bessel sequence in favor of $\mathcal{X}$ by bound μ. We discover a essential circumstance in favor of a sequence $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^*$ to stand an $\mathcal{X}_d$-atomic structure in favor of $\mathcal{X}$ w.r.t a particular operator K if allied sequence space convince subsequent crucial possessions: in support of apiece $\{g_i\}_{i=1}^{\infty}, \{h_i\}_{i=1}^{\infty} \in \mathcal{X}_d$,

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$$\left|\sum_{i=1}^{\infty} g_i h_i\right| \leq \|\{g_i\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \|\{h_i\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \dots\dots (23)$$

For illustration, let $\{g_i\}_{i=1}^{\infty}, \{h_i\}_{i=1}^{\infty} \in \ell_p$ & $p \in (1,2]$. Conjugate of p, q deceit in $[2, \infty)$. Later via Hölder's disparity, sequence ℓ_p on behalf of $1 < p \leq 2$ satisfies (23).

Theorem 2: assent to $\mathcal{X}_d$ stand BK-space. consent to $\{g_i\}_{i=1}^{\infty}$ subsist a structure in $\mathcal{X}^*$ as well as $K \in \mathcal{B}(\mathcal{X}^*)$. condition $\{g_i\}_{i=1}^{\infty}$ be an $\mathcal{X}_d$-atomic structure on behalf of $\mathcal{X}$ w.r.t K and sequence space $\mathcal{X}_d$ contents disparity (23), then here subsists a invariable $\lambda > 0$ s.t.

$$\|K^* f\|_{\mathcal{X}} \leq \lambda \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \dots\dots(24)$$

Proof: Supposing $\{g_i\}_{i=1}^{\infty}$ be $\mathcal{X}_d$-atomic structure on behalf of $\mathcal{X}$ w.r.t K. Then there is particular $c > 0$ s.t. on behalf of each $g \in \mathcal{X}^*$; $a_g = \{a_i\}_{i=1}^{\infty} \in \mathcal{X}_d$ s.t.

$$\|a_g\|_{\mathcal{X}_d} \leq c\|g\|_{\mathcal{X}^*} \dots\dots(25)$$

$K g = \sum_{i=1}^{\infty} a_i g_i$. Later aimed at $f \in \mathcal{X}$,

$$\begin{aligned}
\|K^* f\|_{\mathcal{X}} &= \sup_{g \in \mathcal{X}^*, \|g\|=1} |g(K^* f)| \\
&= \sup_{g \in \mathcal{X}^*, \|g\|=1} |(K g)(f)| \\
&= \sup_{g \in \mathcal{X}^*, \|g\|=1} \left|\sum_{i=1}^{\infty} a_i g_i(f)\right| \qquad \dots.(26) \\
&= \sup_{g \in \mathcal{X}^*, \|g\|=1} \|a_g\|_{\mathcal{X}_d} \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \\
&\leq c \sup_{g \in \mathcal{X}^*, \|g\|=1} \|g\|_{\mathcal{X}^*} \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \left[\|a_g\|_{\mathcal{X}_d} \leq c\|g\|_{\mathcal{X}^*}\right]
\end{aligned}$$

Consequently for a quantity of $c > 0$, $\|K^* f\|_{\mathcal{X}} \leq c\|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d}, \forall f \in \mathcal{X}$.

Definition 5: Consent to $\mathcal{X}$ subsist a Banach space in addition to $\mathcal{X}_d$ stand a BK-space. Consent to $K \in \mathcal{B}(\mathcal{X}^*)$ & $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^*$. We approximately that $\{g_i\}_{i=1}^{\infty}$ be an $\mathcal{X}_d - K$-frame on behalf of $\mathcal{X}$ if subsequent declaration hold:

1. $\{g_i(f)\}_{i=1}^{\infty} \in \mathcal{X}_d$, on behalf of both $f \in \mathcal{X}$;.
2. there exist dual constants s.t.

$$\lambda\|K^* f\|_{\mathcal{X}} \leq \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \leq \mu\|f\|_{\mathcal{X}}, \dots.(27)$$

The lower & upper boundaries of $\mathcal{X}_d$-K-frame are denoted by the elements λ and μ. An $\mathcal{X}_d$- $-I$-frame in favor of X, where I be the individuality operator on $\mathcal{X}^*$, is referred to as an $\mathcal{X}_d$-frame for X. One can think of the collection of $\mathcal{X}_d$-frames in favor of X as a detachment of $\mathcal{X}_d$-K frame in favor of $\mathcal{X}$. For a Banach space $\mathcal{X}$, $\mathcal{X}_d$-K-frame be thus a generalisation of $\mathcal{X}_d$-frame. An example of an $\mathcal{X}_d$ - K-frame that isn't an $\mathcal{X}_d$-frame X is shown below.

Example 2: Consent to $\mathcal{X}$ be space of completely triplets $(\alpha_1, \alpha_2, \alpha_3)$ thru complex scalars and consuming 3/2-norm, signified by $\ell_{3/2}(3)$. Consent to $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^* = \ell_3(3)$ be s.t. $g_i(e_j) = \delta_{ij}$, where e_j 's be vectors in $\mathcal{X}$, consuming 1 in j^{th} lay and 0 away, $g_i = 0\ \forall i \geq 4$. Express $K\colon \mathcal{X}^* \to \mathcal{X}^*$ by

$$Kg_1 = 0, Kg_2 = g_3,\ Kg_3 = g_2 \ldots..(28)$$

Aimed at any $f \in \mathcal{X}$, we require $f = \sum_{i=1}^{3} \alpha_i e_i$

$$\|K^* f\|_{\mathcal{X}} = \|\alpha_2 e_3 + \alpha_3 e_2\|_{3/2} = \left(|\alpha_2|^{3/2} + |\alpha_3|^{3/2}\right)^{2/3} = \|\{g_i(f)\}_{i=2}^{\infty}\|_{\ell_{3/2}} \ldots..(29)$$

Before $\{g_i\}_{i=2}^{\infty}$ be an $\mathcal{X}_d - K$ frame $\mathcal{X}$. Then it be not an $\mathcal{X}_d$-frame as there be no continuous λ s.t. on behalf of some scalar α_1,

$$\lambda\|f\|_{\mathcal{X}} = \left(|\alpha_1|^{3/2} + |\alpha_2|^{3/2} + |\alpha_3|^{3/2}\right)^{2/3} \leq \left(|\alpha_2|^{3/2} + |\alpha_3|^{3/2}\right)^{2/3} = \|\{g_i(f)\}_{i=2}^{\infty}\|_{\ell_{3/2}} \ldots\ldots(30)$$

Proposition 4: If $\{g_i\}_{i=1}^{\infty}$ stands an $\mathcal{X}_d$-frame on behalf of $\mathcal{X}$ in addition to $K \in \mathcal{B}(\mathcal{X}^*)$, before $\{Kg_i\}_{i=1}^{\infty}$ be an $\mathcal{X}_d - K$-frame in support of $\mathcal{X}$.

Proof: Assume $\{g_i\}_{i=1}^{\infty}$ be an $\mathcal{X}_d$ frame $\mathcal{X}$, $\{g_i(f)\}_{i=1}^{\infty} \in \mathcal{X}_d$, on behalf of all $f \in \mathcal{X}$ and here stand constants $0 < \lambda \leq \mu < \infty$ s.t. on behalf of apiece

$$\lambda\|f\|_{\mathcal{X}} \leq \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \leq \mu\|f\|_{\mathcal{X}} \ldots\ldots(31)$$

Let $f \in \mathcal{X}$ exist fixed. Later $(Kg_i)(f) = g_i(K^* f)$ in addition to K^*, we involve $\{(Kg_i)(f)\}_{i=1}^{\infty} \in \mathcal{X}_d$. Similarly, $\|K^* f\|_{\mathcal{X}} \leq \|K\|\|f\|_{\mathcal{X}}$ affords that on behalf of $f \in \mathcal{X}$,

$$\lambda\|K^* f\|_{\mathcal{X}} \leq \|\{(Kg_i)(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \leq \mu\|K\|\|f\|_{\mathcal{X}} \ldots..(32)$$

So $\{Kg_i\}_{i=1}^{\infty}$ be an $\mathcal{X}_d - K$ frame for $\mathcal{X}$. A $\mathcal{X}_d$-Bessel sequence be an $\mathcal{X}_d$ - K-frame, as the example below shows, but this be not case for the further operator T.

Example 3: Let $\mathcal{X} = \ell_{3/2}(3)$. Consent to $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^* = \ell_3(3)$ subsist s.t. for $i = 1, 2, 3,\ g_i(e_j) = \delta_{ij},\ g_i = 0\ \forall\ i \geq 4$. Outline K as well as T as of $\mathcal{X}^*$ to $\mathcal{X}^*$ as shadows:

$$K\ g_1 = 0, K\ g_2 = g_3, K\ g_3 = g_2$$

$$T\ g_1 = g_1, T\ g_2 = g_3, T\ g_3 = g_2$$

Before $\{g_i\}_{i=2}^{\infty}$ be an $\mathcal{X}_d - K$ frame but it stands not an $\mathcal{X}_d - T$-frame in favor of $\mathcal{X}$.

Theorem 3: Consent to $\{g_i\}_{i=1}^{\infty}$ subsist an $\mathcal{X}_d$-K-frame in favor of $\mathcal{X}$. Let $T \in \mathcal{B}(\mathcal{X}^*)$ be s.t. $R(T) \subseteq R(K)$. Subsequently $\{g_i\}_{i=1}^{\infty}$ stands an $\mathcal{X}_d - T$-frame on behalf of $\mathcal{X}$.

Proof: Expect $\{g_i\}_{i=1}^{\infty}$ stands an $\mathcal{X}_d$ - K-frame on behalf of $\mathcal{X}$. There is constants $0 < \lambda \leq \mu < \infty$ s.t. on behalf of both $f \in \mathcal{X}$

$$\lambda \|K^* f\|_{\mathcal{X}} \leq \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \leq \mu \|f\|_{\mathcal{X}} \quad \ldots.(33)$$

we obligate for both $f \in \mathcal{X}$

$$\frac{\lambda}{c} \|T^* f\|_{\mathcal{X}} \leq \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \leq \mu \|f\|_{\mathcal{X}} \ldots\ldots.(34)$$

Later $\{g_i\}_{i=1}^{\infty}$ be an $\mathcal{X}_d - T$-frame on behalf of $\mathcal{X}$.

5. Conclusion

Frames on behalf of operators are introduced on behalf of a set of incessant linear functionals built on a Banach space. Results on the generation of frames on behalf of operators are presented, together by the sufficient and required situation that needs to be satisfied. In addition, it be exposed that the notions of "atomic structure" and "frame for machinist" as used in the Paper is not generally the same, if the associated sequence spaces violate some additional requirements.

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