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MODULARITY FRAMEWORK OF PARTITION THEORY BY INVESTIGATING GALOIS REPRESENTATIONS AND MODULAR CURVES FOR EXTENSIONS OF KNOWN CONGRUENCES

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Abstract: Once again, the question of whether these new families could be described combinatorially emerged. Since it fulfills precisely the same type of congruences as those found by Ono for the partition function, one of the main conclusions of this thesis proves that the same crank established by Andrews-Garvan is a sort of "universal" statistic for partition congruences. The primary finding in this case is the existence of positive density sets of primes ℓ for which nearly all of these quotients' coefficients disappear modulo powers of ℓ. These findings highlight two important applications of modular forms in number theory, and the techniques used here easily generalize to more general classes of modular forms. The analytic modular can be used to effectively understand the coefficients of these and many other number-theoretic functions.

Keywords: partition functions, modular curves, **Congruences,** partition theory, modular forms, restricted partition

1. INTRODUCTION:

The classical partition function counts all of the different ways in which a positive integer can be expressed as a sum of smaller positive integers. In contrast, restricted partition functions only consider partitions that meet certain conditions or constraints. The limitations that are frequently imposed on partitions include restrictions on the size, kind, or multiplicity of the portions that constitute the partition. Restricted partitions are an important area of research because they reveal more profound combinatorial and arithmetic patterns that are not evident in unrestricted partitions. In addition, numerous natural issues in the fields of mathematics and physics require such limitations. One of the most interesting and remarkable elements of partition theory is the presence of congruence phenomena, which relate to the regular and predictable patterns that may be observed in partition numbers when they are analyzed modulo integers. In contrast to basic counting, which enumerates the number of partitions of a given integer, congruences reveal deeper arithmetic features and underlying symmetries in the structure of partitions. The earliest observations and systematic studies of these phenomena took place in the early twentieth century. Srinivasa Ramanujan was the most prominent figure in the initial observations and systematic studies of these phenomena, and these phenomena have since become a central focus of research in the field of number theory.

The Ramanujan congruences for the unrestricted partition function are among the simplest and most well-known examples of congruences. Ramanujan demonstrated that the number of partitions of specific integers is divisible by small prime numbers in a very systematic manner. For example, he discovered that the partition numbers adhere to patterns such as the following: the number of partitions of integers that can be expressed as 5n+4 is divisible by 5, the number of partitions of integers that can be expressed as 7n+5 is divisible by 7, and similarly for 11. This is true for all non-negative integers n. The findings were extraordinary due to the fact that they demonstrated that partition numbers, which appear irregular when presented in a sequence, really adhere to exact arithmetic laws when considered under modular arithmetic.

Congruence phenomena are also demonstrated by partition functions that have been restricted. The imposition of limitations such as distinct parts, odd parts, or bounded parts can result in sequences of partition numbers that frequently exhibit congruences that reflect or generalize those that are found in the unrestricted situation. For example, the number of partitions into separate parts or into odd parts may satisfy congruences modulo small primes, indicating hidden arithmetic structures that are not immediately obvious from direct enumeration. It is possible to gain insight into the way in which combinatorial limits affect arithmetic behavior by studying these congruences, and this research also draws attention to the relationship between number theory and combinatorics.

Congruences in partition functions are investigated in close conjunction with modular forms and generating functions. Generating functions enable mathematicians to manipulate partition numbers algebraically or analytically by encoding them in series. As soon as these generating functions display modular features, they become formidable instruments for deriving congruences. Modular forms are highly symmetric functions that have profound mathematical significance. Mathematicians are able to provide explanations, generalizations, and predictions for congruences for both unrestricted and confined partitions by using modular forms. The study of congruences was turned from a collection of discrete numerical data into a systematic and theoretically grounded field as a result of this modular perspective.

Congruence events are not only theoretically elegant; they also have important consequences for broader fields of mathematics. They provide linkages between partition theory and elliptic functions, q-series, and representation theory. Furthermore, the study of congruences has been the source of inspiration for a wide range of research in modern number theory. This research has led to the creation of novel techniques that have been used to uncover infinite families of congruences, as well as their applications in combinatorial identities, algebraic geometry, and even cryptography.

In modern number theory, modular forms have an equal number of important applications. Early developments in the field include the study of theta functions, generating functions for class numbers, representations by quadratic forms, analytic features modular L-functions, and elliptic functions. These disciplines have dominated number theory research since the 19th century and continue to influence current research in the subject. Modular forms have continued to play a fundamental role in the proofs of many groundbreaking results in recent years, including Wiles's proof of Fermat's Last Theorem and the works of Fields medalists Borcherds and Deligne on the connections between the coefficients of modular forms and Galois representations. The arithmetic of the coefficients of modular forms is crucial because in many of these contexts, modular forms emerge as the generating functions of number theoretic functions.

In general, a function on the complex upper-half plane that satisfies specific analytic transformation criteria and possesses a Fourier series is called a modular form. The ability of modular transformation laws to "translate" the function's analytic features into striking combinatorial correlations among the coefficients is what gives modular forms their strength in number theory applications. This thesis's findings build on some of the author's earlier publications that emphasize the interaction between two well-known examples of modular forms' analytic and combinatorial characteristics. A search-based multi-view clustering method was presented by Saeidi et al. (2015) for the analysis and evolution of large-scale software systems. The increasing demand for scalable methods in software architecture, especially in complicated and legacy systems, was the focus of this study. By studying fake theta functions and weakly holomorphic modular forms modulo 2 and 3, Ahlgren and Kim (2015) made important contributions to number theory.

Ahlgren and Kim (2015) studied weakly holomorphic modular forms modulo 2 and 3 and mimic theta functions to examine the modular behavior of mathematical objects. In his doctoral dissertation, Wang (2017) examined the arithmetic characteristics of partitions and deduced a number of Hecke-Rogers-type identities. In a similar vein, Kim (2010) used q-combinatory to study the arithmetic features of partition functions, providing new identities and congruence that emphasized the nuanced relationships between partition theory and modular forms. Congruence for constrained bipartitions modulo powers of five was studied by Wang (2017). His results showed the significance of q-series factorization for a more detailed understanding of congruence behavior. Partition functions in 2D CFTs, or flavored 2D conformal field theories, were studied by Dyer et al. (2018).

In conclusion, partition functions that exhibit congruence phenomena demonstrate the presence of complex arithmetic and modular structures that are embedded within problems that may appear to be simple counting problems. The congruences emphasize the complex interactions that exist between combinatorics, number theory, and analysis. These interactions are illustrated by the pioneering discoveries of Ramanujan and more recent explorations into confined partitions and modular forms. Congruences continue to be a fundamental and active topic of research, and they provide insight into the patterns and hidden symmetries of integers and their partitions.

2. OBJECTIVES OF THE STUDY

- **To investigate how cusp forms, Eisenstein series, and eta-products affect limited partition generating function congruence.**
- To find the modularity framework of partition theory by investigating galois representations and modular curves for extensions of known congruences

3. NEED OF THE STUDY

Partition functions are crucial in number theory and combinatorics, and their congruence properties—especially those discovered by Ramanujan—have sparked extensive research. The congruence behavior of limited partition functions, such as those with odd or different components or specific modular constraints, is poorly understood, particularly when viewed from the perspective of modular forms, despite the fact that classical partition functions have been thoroughly studied. Additionally, existing methods are either purely theoretical or computational and frequently lack adequate integration between theory and computers. To address this problem, the current study investigates whether new Ramanujan-type Congruences can be generated for wider classes of restricted partitions and how congruence relations are controlled by modular properties of generating functions.

4. RESEARCH METHODOLOGY

Theoretical Framework and Mathematical Foundations: The investigation will begin with a thorough review of the literature on partition theory, modular forms, and integer congruences. Important works by Ramanujan, Atkin, Serre, and Ono as well as more recent developments in the arithmetic of modular forms will be thoroughly examined to offer a solid mathematical basis. Definitions of limited partition functions, such as those with odd parts, separate pieces, or congruence criteria, will be formalized. Basic concepts in the theory of modular forms, q-series, eta-quotients, and Hecke algebras will also be studied in order to offer a strong conceptual basis for deriving Congruences.

Computational Experimentation and Verification: Because of the complexity of modular forms and the behavior of their coefficients, computational approaches will be crucial to the study. Symbolic algebra programs such as Sage Math, Mathematica, and PARI/GP will be used to compute and analyze large sequences of the partition functions' coefficients. These computer experiments will help identify potential congruences and validate theoretical predictions. By identifying numerical patterns, candidate congruences will be hypothesized and subsequently mathematically codified. Cross-validation will be carried out utilizing established Congruences from the literature as benchmarks to ensure accuracy and reliability.

Comparative and Generalization Studies: This will help determine whether any unifying modular rules apply to these congruences. Recently discovered congruences will be extended whenever possible to more general classes of partitions or to moduli larger than small primes. It is also possible to look into potential connections with harmonic Maass forms, Galois representations, or fictitious modular forms. The goal of this more thorough modular method is to increase the long-term mathematical significance and theoretical depth of the research.

5. DATA ANALYSIS AND RESULTS

One of the most effective approaches in contemporary number theory is the theory of modular forms. It also serves an important role in helping to comprehend the arithmetic features of partition functions. Modular forms are complex functions that exhibit a high degree of symmetry. They meet particular transformation features under the action of modular groups. They have exceptional analytic, algebraic, and arithmetic structures, which make them excellent for examining congruences, generating functions, and the profound relationships that exist within number theory. It is primarily via the use of generating functions that the relationship between partition functions and modular forms comes into existence. A generating function encodes a sequence of numbers—such as the number of partitions of integers—into a power series, where the exponent of the variable tracks the size of the partition and the coefficient represents the number of partitions of that size. Modularity or near-modularity is exhibited by several generating functions that are related with partitions, particularly the unrestricted partition function. This property allows mathematicians to apply the rich machinery of modular forms to study partitions, including deriving asymptotic formulas, proving congruences, and uncovering hidden symmetries. Analytic methods were employed to analyze the growth of partition numbers by Hardy and Ramanujan, two mathematicians who are considered as having established one of the oldest and most influential linkages. Later on, mathematicians discovered that the generating function of the unrestricted partition function may be stated as an infinite product that exhibits modular features. This insight was essential in providing an explanation for the classical Ramanujan congruences (mod 5, 7, 11) and provided the foundation for expanding congruence results to generalized and restricted partitions. The congruences of Theorem provide certain arithmetic progressions such that $a_i(n) \equiv 0(\mathrm{mod}\,\ell^a)$, where $\ell \in \{3,7\}$ and a is a small, positive integer. Berndt and Yee's proof relies on the fact that each of the series has the form

$$F(q) = 1 + M \sum_{n=1}^{\infty} a(n)q^n \ldots.(1)$$

where M is some rational number. Setting $G(q) := 1/F(q)$ and observing that $G(q)F(q) = 1$ leads to a simple functional equation for the power series of $G(q)$, namely

$$G(q) = 1 - G(q)\left(M \sum_{n=1}^{\infty} a(n) q^n\right) \ldots .(2)$$

This can then be iterated to give an " M-adic" expansion

$$\begin{aligned} G(q) &= 1 - M \sum_{n_1=1}^{\infty} a(n_1) q^{n_1} + M^2 \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} a(n_1) a(n_2) q^{n_1+n_2} + \cdots \\ &= 1 + \sum_{k=1}^{\infty} (-1)^k M^k \sum_{n_1,\ldots,n_k=1}^{\infty} a(n_1) \cdots a(n_k) q^{n_1+\cdots+n_k} \ldots (3) \end{aligned}$$

The generating function for overpartitions is

$$\sum_{n\geq 0} \bar{p}(n) = \prod_{n\geq 1} \frac{1+q^n}{1-q^n} = \left(1 + 2\sum_{n\geq 0} (-1)^n q^{n^2}\right)^{-1}$$

This means that there is an inverse series expansion of the form (1) with $M = 2$. The density then follows from asymptotics and parity properties for the number of representations of integers by simple diagonal quadratic forms.

The existence of $F_m(z)$

Lemma: If τ is sufficiently large, then there is some $\lambda \geq 1$ such that

$$\frac{\widetilde{G_m}(24z)}{\eta^{\ell}(24\ell z)} \cdot E_{j+1}(24z)^{\ell^{\tau}} \in S_{\lambda+\frac{1}{2}}\left(\Gamma_1(576\ell^2 N^2)\right) \ldots .(4)$$

Proof: Basic facts about modular forms along with Proposition, and (3) imply that $\widetilde{G_m}(24z) \in M^{!}_{(\ell+1)/2}\left(\Gamma_1(48\ell^2 N^2)\right)$. Combining this with the eta-product factors makes it clear that $\Gamma_1(576\ell^2 N^2)$ is the appropriate congruence subgroup, we know that $E_{j+1}(z)$ vanishes at each cusp $\frac{a}{c}$ where $\ell N \nmid c$. Once τ is taken to be sufficiently large, it only remains to be shown that $\widetilde{G_m}(z)/\eta^{\ell}(\ell z)$ vanishes at each cusp $\frac{a}{c}$ with $\ell N \mid c$, as replacing z by $z/24$ does not affect the signs of the cusp orders. Such a cusp $\frac{a}{c}$ is associated with a matrix of the form

$$A_{a,c} := \begin{pmatrix} \bar{a} & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \in \Gamma_0(\ell N),,\ldots(5)$$

since $e^{2\pi i (A_{a,c} z)}$ vanishes as $z \to \frac{a}{c}$. The expansion of the denominator function at the cusp $\frac{a}{c}$ with $\ell N \mid c$ is

$$\left.\frac{1}{\eta^{\ell}(\ell z)}\right|_{\ell/2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = q^{\ell^2/24} + \cdots (6)$$

Thus the proof of the lemma hinges on showing that the expansion of $\widetilde{G_m}(z)$ at $\frac{a}{c}$ is $(*\, q^h + \cdots)$ for some $h > \ell^2/24$

$$G_m(z)|_{\frac{\ell+1}{2}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{-i}{2\pi}\left(\sum_{s=1}^{N-1} \frac{\eta^\ell(\ell z)}{\eta(z)} \cdot \frac{\omega_s \zeta^{-ms}}{t_{0,s}(z)}\right)\Bigg|_{\frac{\ell+1}{2}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \frac{-i}{2\pi}\left(\frac{d}{\ell}\right)\frac{\eta^\ell(\ell z)}{\eta(z)}\sum_{s=1}^{N-1} \frac{\omega_s \zeta^{-ms}}{\beta_s t_{0,\overline{ds}}(z)} \dots.(7)$$

$$t_{0,s}(z)\big|_{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \beta_s \cdot t_{0,\overline{ds}}(z) \dots(8)$$

To find the expansion of $G_m(z) \otimes \left(\frac{\cdot}{\ell}\right)$, first observe that for any $v' \equiv d^2 v (\mathrm{mod}\ell)$ there is a commutation relation

$$\begin{pmatrix} 1 & -v/\ell \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix} \dots.(9)$$

where

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} := \begin{pmatrix} a - cv/\ell & b - cvv'/\ell^2 + (av' - dv)/\ell \\ c & d + cv'/\ell \end{pmatrix} \in \Gamma_0(\ell N) \dots(10)$$

set $g := g_\ell$

$$\left(G_m(z) \otimes \left(\frac{\cdot}{\ell}\right)\right)\Bigg|_{\frac{\ell+1}{2}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \dots(11)$$

$$= \frac{g}{\ell}\sum_{v=1}^{\ell-1}\left(\frac{v}{\ell}\right) G_m(z) \left|\begin{pmatrix} 1 & -\frac{v}{\ell} \\ 0 & 1 \end{pmatrix}\right|_{\frac{\ell+1}{2}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \dots.(12)$$

$$= \frac{g}{\ell}\sum_{v=1}^{\ell-1}\left(\frac{v}{\ell}\right) G_m(z) \Bigg|_{\frac{\ell+1}{2}}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\Bigg| \begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix} \dots(13)$$

$$= \frac{-ig}{2\pi\ell}\sum_{v=1}^{\ell-1}\left(\frac{v}{\ell}\right)\left(\sum_{s=1}^{N-1} \frac{\eta^\ell(\ell z)}{\eta(z)} \cdot \frac{\omega_s \zeta^{-ms}}{t_{0,s}(z)}\right)\Bigg|_{\frac{\ell+1}{2}}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\Bigg| \begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix} \dots(14)$$

Now this can be evaluated using the modular transformation properties of $t_{0,s}(z)$ and $\eta^\ell(\ell z)/\eta(z)$.

$$\left(G_m(z) \otimes \left(\frac{\cdot}{\ell}\right)\right)\Big|_{\frac{\ell+1}{2}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \dots (15)$$

$$= \frac{-ig}{2\pi\ell}\left(\frac{d'}{\ell}\right)\sum_{s=1}^{N-1}\sum_{v=1}^{\ell-1}\left(\frac{v}{\ell}\right)\frac{\eta^{\ell}(\ell z)}{\eta(z)}\cdot\frac{\omega_s\zeta^{-ms}}{\beta_s' t_{0,\overline{d's}}(z)}\Bigg| \begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix} \dots (16)$$

Thus the expansion of (16) begins as

$$\frac{-i}{2\pi}\left(\frac{d}{\ell}\right)\left(\sum_{s=1}^{N-1}\frac{\omega_s\zeta^{-ms}}{\omega_{\overline{ds}}\cdot\beta_s}q^{\delta_\ell}+\cdots\right)\dots.(17)$$

and (17) begins with

$$\frac{-ig}{2\pi\ell}\left(\frac{d}{\ell}\right)\left(\sum_{s=1}^{N-1}\frac{\omega_s\zeta^{-ms}}{\omega_{\overline{ds}}\cdot\beta_s}\sum_{v=1}^{\ell-1}\left(\frac{v}{\ell}\right)q^{\delta_\ell}\Bigg|\begin{pmatrix} 1 & -v'/\ell \\ 0 & 1 \end{pmatrix}+\cdots\right)\dots(18)$$

$$= \frac{-ig}{2\pi\ell}\left(\frac{d}{\ell}\right)\left(q^{\delta_\ell}\sum_{s=1}^{N-1}\frac{\omega_s\zeta^{-ms}}{\omega_{\overline{ds}}\cdot\beta_s}\sum_{v=1}^{\ell-1}\left(\frac{v}{\ell}\right)e^{-2\pi i v'\delta_\ell/\ell}+\cdots\right)..(19)$$

Multiplying the first term of the above equation by $-\epsilon_\ell$ gives precisely the negative of the displayed term in (17), and hence they cancel in $\widetilde{G_m}(z)$. The cusp expansion has only integral powers of q due to the series expansion of the Klein forms and eta-quotient in (18) and (19), and therefore must have the form $(* q^{\delta_\ell+1} + \cdots)$. Since $\delta_\ell + 1 > \ell^2/24$, the proof is complete.

6. CONCLUSION

The significance of the arithmetic of modular forms in number theory is exemplified by the divisibility features of the partition function and the coefficients of Eisenstein series, and the findings of this thesis demonstrate the continued advancement of this area. It is not unexpected that many of the modular forms that originate as generating functions of natural number-theoretic objects live in spaces of modular forms with fewer analytic constraints because holomorphic modular forms are very special analytic objects. Nonetheless, these functions' l-adic characteristics are frequently still rather pleasant, and since the theory of l-adic modular forms also covers a larger class of functions, it is now easier to examine the arithmetic of their coefficients. Furthermore, the relationship that exists between partitions, modular forms, and number theory offers a framework that allows for the discovery of universal patterns in the fields of arithmetic and combinatorics. When mathematicians embed discrete sequences into modular frameworks, they can uncover similarities between objects that at first glance appear to be unconnected, make predictions about new congruences, and investigate profound links across various mathematical disciplines. This comprehensive approach serves to illustrate that the study of partitions is not a standalone combinatorial undertaking but rather a route to deeper mathematical insights. The study of partitions has the potential to affect both theoretical research and applications in a variety of mathematical domains.

7. REFERENCES

- Ahlgren, S., and Kim, B. (2015). Mock theta functions and weakly holomorphic modular forms modulo 2 and 3. *In Mathematical Proceedings of the Cambridge Philosophical Society, Cambridge University Press,* 158(1), 111-129.
- Ahlgren, S., and Kim, B. (2015, January). Mock theta functions and weakly holomorphic modular forms modulo 2 and 3. In Mathematical Proceedings of the Cambridge Philosophical Society. *Cambridge University Press,* 158(1), 111-129.
- Dyer, E., Fitzpatrick, A. L., and Xin, Y. (2018). Constraints on flavored 2d CFT partition functions. *Journal of High Energy Physics,* (2), 1-37.
- Holt, T. J., & Bossler, A. M. (2008). Examining the applicability of lifestyle-routine activities theory for cybercrime victimization. *Deviant behavior*, *30*(1), 1-25.
- Matta, F. K., Scott, B. A., Koopman, J., & Conlon, D. E. (2015). Does seeing “eye to eye” affect work engagement and organizational citizenship behavior? A role theory perspective on LMX agreement. *Academy of Management Journal*, *58*(6), 1686-1708.
- Xiao, S., & Li, Y. (2012). Modeling and high dynamic compensating the rate-dependent hysteresis of piezoelectric actuators via a novel modified inverse Preisach model. *IEEE Transactions on Control Systems Technology*, *21*(5), 1549-1557.
- Muniz, M. N., and Oliver-Hoyo, M. T. (2014). Investigating quantum mechanical tunneling at the nanoscale via analogy: Development and assessment of a teaching tool for upper-division chemistry. *Journal of Chemical Education*, 91(10), 1546-1556.
- Wang, L. (2017). Arithmetic properties of partitions and Hecke-Rogers type identities (Doctoral dissertation, National University of Singapore), 1-78.