

Effect of Bi-Parabolic Temperature Variation on the Vibration of Isotropic Viscoelastic Square Plates (S-F-S-F) with Circular Thickness Variation

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Abstract : This paper investigates the influence of bi-parabolic temperature variation on the free vibration behavior of a viscoelastic square plate with circular thickness variation, subjected to simply supported–free–simply supported–free boundary conditions. This topic is particularly important in the design of structures used across various industries, including civil, mechanical, automotive, and aerospace engineering.

The study employs thin plate theory to analyze the vibrational characteristics of the plate, taking into account the effects of temperature variation and circular thickness distribution. The primary objective is to examine the vibration of the square plate along both the x and y directions while incorporating thermal effects. To solve the vibration equation, the Rayleigh–Ritz method is utilized, and the first two modes of vibration are evaluated.

Keywords: Vibrations, isotropic, square plate, simply supported-free edges, biparabolic temperature, circular thickness.

1.1 Introduction :

Vibration plays a crucial role in the design of mechanical and engineering structures. As technology continues to advance, understanding the impact of temperature on viscoelastic plates of various dimensions has become more important. Simply supported or clamped plates exposed to different thermal conditions are widely used in industries such as automotive, aerospace, power generation, and marine engineering. A number of studies have investigated how temperature influences the vibration behavior of both homogeneous and heterogeneous plates with varying thicknesses. In this study, the focus is on examining how different material variations affect the vibration characteristics of a square plate.

Leissa authored a detailed monograph on the vibration behavior of various plate types, offering extensive numerical data on recurrence parameters for different boundary conditions. **Narinder**

Kaur investigated the impact of thermal gradients on the vibration of triangular plates, assuming bilinear thickness tapering and a linear temperature gradient in one direction. **Gorman** presented an analytical approach for studying the vibration of simply supported right triangular plates, focusing on validating eigenvalues for the first four vibration modes. His method is adaptable to right triangular plates with varying boundary conditions. Using a finite element method, **Mirza and Bijlani** examined the vibration of cantilevered triangular plates with variable thickness. They also analyzed natural frequencies and mode shapes for various combinations of four non-dimensional geometric parameters, particularly aspect ratios and thickness ratios. The frequencies were categorized across different cases, with representative mode shapes illustrated for select configurations. **Gorman** further studied the vibration of right triangular plates under all possible clamped boundary conditions, offering detailed results for mode shapes and frequencies. Using a modified superposition method, he also evaluated the natural frequencies and mode shapes for the first four vibration modes of plates with a wide range of aspect ratios. **Leissa and Jaber** employed the Ritz method to perform a comprehensive study of free vibration in triangular plates. They modeled displacement functions using algebraic polynomials and presented the first six natural frequencies and nodal patterns for 17 different triangular plate configurations, obtained by varying side length ratios. **Liew and Chiam** explored free vibration of isotropic and symmetrically laminated composite plates from the generic triangular plate using the Rayleigh – Ritz approach. The scientists looked at how the plate's natural frequencies were affected by geometry, material qualities, and lamination. **Singh and Hassan** utilised the Rayleigh – Ritz approach to develop numerical solutions to the vibration issue of triangular plates with uniform and arbitrarily changing thicknesses for varied boundary conditions. **Sakiyama and Haung** demonstrated a free vibration analysis of a right triangular plate with various thicknesses and boundary conditions. The authors discovered that the triangular plate was a non-uniformly thick rectangle plate. Based on an exact theory of three-dimensional elasticity, **Cheung and Zhou** investigated free vibration of triangular plates with cantilevered and entirely free isosceles. From the strain energy and the kinetic energy of the plate, the Ritz approach is utilized to get the equation for one's own frequency. **Chakarvarty** offered a wealth of data on the vibration qualities of several plate types under varied boundary circumstances. The author presented accurate data for linear vibration of elastic plates under various boundary conditions. **Zhang and Li** proposed a method for analyzing the vibration of randomly produced triangular plates with elastically bonded edges. The displacement function is written as a two-dimensional Fourier cosine series, with additional one-dimensional series added to increase the convergence and accuracy of the displacement solution. **Chaudhary and Falak** published a paper on various materials on a laminated triangular plate with free-clamped boundary conditions. The study was carried out on the isotropic right triangular plate and the symmetrically laminated / composite triangular plate. The impact of different physical and geometric characteristics on the

natural frequencies of the triangular equilateral plate exposed to classical boundary conditions were studied by **Pradhan and Chakraverty**. To acquire the problem of own value in question, the numerical modelling is done using the Rayleigh–Ritz approach. In both tabular and visual representations, **Khanna and Kaur** explored the first two modes of natural frequency parameters for the vibration of a tapered rectangular visco-elastic isotropic plate under various heat circumstances. The Rayleigh–Ritz approach was used to investigate the influence of non-homogeneity on free vibrations of a rectangular plate. **Sharma and Bensal** studied the Effect of vibration on tapered triangular plate with simply supported boundary condition under thermal condition

This research focuses on the Effect of Bi-parabolic temperature variation on the vibration of isotropic visco-elastic square plate (S-F-S-F) with circular thickness. Values for different parameters like non-homogeneity, temperature gradient and thickness variations determines the vibration frequencies of the first and second modes.

1.2 Transformation of Square Plate :

1.2.1 Geometry :

Let's look about the x , y , and z illustrated in Figure 1.1, which together define the visco-elastic square plate:

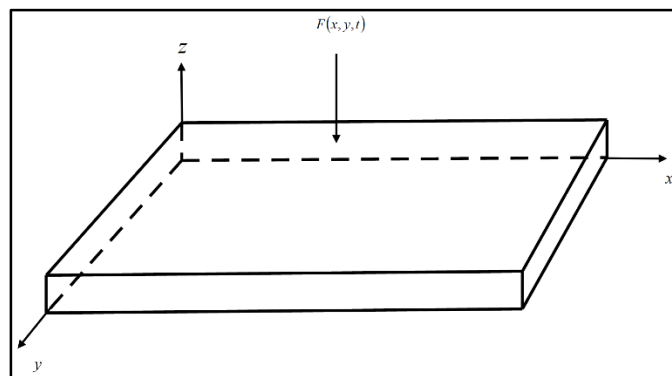


Fig: 1.1: Geometry of Square Plate

1.3 Analysis of Equation of motion :

Using rectangular coordinates, the differential equation of motion for an isotropic square plate is:

$$\tilde{D} \left[\begin{aligned} &D_1 (w_{xxxx} + 2w_{xyxy} + w_{yyyy}) + 2D_{1_x} (w_{xxx} + w_{xyy}) + 2D_{1_y} (w_{yyy} + w_{yxx}) \\ &+ D_{1_{xx}} (w_{xx} + \nu w_{yy}) + D_{1_{yy}} (w_{yy} + \nu w_{xx}) + 2(1-\nu)D_{1_{xy}} w_{xy} \end{aligned} \right] + \rho h w_{tt} = 0 \quad (1.1)$$

Multiplying the deflection function by the time function yields as :

$$w(x, y, t) = W(x, y) \times T(t) \quad (1.2)$$

Where the deflection function in x and y is $W(x, y)$, and $T(t)$ is a time function.

Incorporating (1.2) into (1.1), we get

$$\left[\begin{aligned} &D_1 (W_{xxxx} + 2W_{xyxy} + W_{yyyy}) + 2D_{1_x} (W_{xxx} + W_{xyy}) + 2D_{1_y} (W_{yyy} + W_{yxx}) \\ &+ D_{1_{xx}} (W_{xx} + \nu W_{yy}) + D_{1_{yy}} (W_{yy} + \nu W_{xx}) + 2(1-\nu)D_{1_{xy}} W_{xy} \end{aligned} \right] / \rho h W = -T_{tt} / \tilde{D} T \quad (1.3)$$

If both sides of the equation (1.3) are equal to the constant p^2 , then the equation is true,

$$\left[\begin{aligned} &D_1 (W_{xxxx} + 2W_{xyxy} + W_{yyyy}) + 2D_{1_x} (W_{xxx} + W_{xyy}) + 2D_{1_y} (W_{yyy} + W_{yxx}) \\ &+ D_{1_{xx}} (W_{xx} + \nu W_{yy}) + D_{1_{yy}} (W_{yy} + \nu W_{xx}) + 2(1-\nu)D_{1_{xy}} W_{xy} \end{aligned} \right] - \rho p^2 h W = 0 \quad (1.4)$$

and

$$T_{tt} + p^2 \tilde{D} T = 0 \quad (1.5)$$

The differential equation of motion and the time function for a square plate are shown in equations (1.4) and (1.5), respectively.

For this context, D_1 represents the flexural stiffness of a plate defined as:

$$D_1 = \frac{Eh^3}{12(1-\nu^2)} \quad (1.6)$$

where E is the elasticity modulus, ν is the Poisson's ratio.

Next, the Rayleigh-Ritz technique is used to these differential equations to determine the frequency parameter.

1.4 Mathematical Assumptions :

It is assumed that temperature varies parabolically in two directions i.e.

$$\tau = \tau_0(1 - x^2/a^2)(1 - y^2/a^2) \quad (1.7)$$

where τ denotes the temperature excess above the reference temperature at any point on the plate and τ_0 denotes the temperature at any point on the boundary of plate and “a” is the length of a side of square plate. The temperature dependence of the modulus of elasticity for most of engineering materials can be expressed in this

$$E = E_0(1 - \gamma\tau) \quad (1.8)$$

where, E_0 is the value of the Young's modulus at reference temperature i.e. $\tau = 0$ and γ is the slope of the variation of E with τ . The modulus variation (1.10) become

$$E = E_0[1 - \alpha(1 - x^2/a^2)(1 - y^2/a^2)] \quad (1.9)$$

where $\alpha = \gamma\tau_0$ ($0 \leq \alpha < 1$), thermal gradient.

It is assumed that thickness also varies bi-linear in x and y directions as shown below:

$$h = h_0 \left(1 + \left[1 + \beta_1 \left(1 - \sqrt{1 - \frac{x}{a}} \right) \right] \left[1 + \beta_2 \left(1 - \sqrt{1 - \frac{y}{a}} \right) \right] \right) \quad (1.10)$$

where β_1 is taper parameters in x- directions respectively and $h=h_0$ at $x=y=0$. Put the value of E & h from equation (1.9) & (1.10), one obtain

$$D_1 = \frac{[E_0[1 - \alpha(1 - x/y)(1 - y/x)]h_0(1 + \beta_1(1 - \sqrt{1 - \frac{x}{a}}))(1 + \beta_2(1 - \sqrt{1 - \frac{y}{a}}))]}{12(1 - \nu^2)} \quad (1.11)$$

Rayleigh-Ritz technique is applied to solve the frequency equation. In this method, one requires maximum strain energy must be equal to the maximum kinetic energy. So it is necessary for the problem under consideration that

$$\delta(K^* - S^*) = 0 \quad (1.12)$$

for arbitrary variations of W satisfying relevant geometrical boundary conditions. Since the plate is assumed as clamped at all the four edges, so the boundary conditions are:

$$\left. \begin{aligned} W = W_{,x} = 0, & \quad x = 0, a \\ W = W_{,y} = 0, & \quad y = 0, a \end{aligned} \right\} \quad (1.13)$$

Now assuming the non-dimensional variables as

$$X = \frac{x}{a}, \quad Y = \frac{y}{a}, \quad \bar{W} = \frac{W}{a}, \quad \bar{h} = \frac{h}{a} \quad (1.14)$$

The two-term deflection function is taken as :

$$W = \left(\frac{x}{a}\right)\left(1 - \frac{x}{a}\right) \left[A_1 + A_2 \left(\frac{y}{a}\right) \left(1 - \frac{y}{a}\right) \right] \quad (1.15)$$

In the above equation A_1 and A_2 are constants satisfy boundary conditions.

The kinetic energy K^* and strain energy S^* are :

$$K^* = \left(\frac{1}{2}\right) \rho p^2 \bar{h}_0 a^5 \int_0^1 \int_0^1 [(1 + \beta_1(1 - \sqrt{1 - X}))(1 + \beta_2(1 - \sqrt{1 - Y}))\bar{W}^2] dYdX \quad (1.16)$$

and

$$S^* = Q \int_0^1 \int_0^1 \left[[1 - \alpha(1 - X^2)(1 - Y^2)] [(1 + \beta_1(1 - \sqrt{1 - X}))(1 + \beta_2(1 - \sqrt{1 - Y}))\sqrt{1 - Y}] \right]^3 \left\{ (\bar{W}_{,XX})^2 + (\bar{W}_{,YY})^2 + 2\vartheta \bar{W}_{,XX} \bar{W}_{,YY} + 2(1 - \vartheta)(\bar{W}_{,XY})^2 \right\} dYdX \quad (1.17)$$

where, $Q = \frac{E_0 h_0^3 a^3}{24(1 - \vartheta^2)}$

Using equations (1.16) & (1.17) in equation (1.12), one get

$$(S^{**} - \lambda^2 K^{**}) = 0 \quad (1.18)$$

where,

$$S^{**} = \int_0^1 \int_0^1 \left[[1 - \alpha(1 - X^2)(1 - Y^2)] (1 + \beta_1(1 - \sqrt{1 - X}))(1 + \beta_2(1 - \sqrt{1 - Y})) \left\{ (\bar{W}_{,XX})^2 + (\bar{W}_{,YY})^2 + 2\vartheta \bar{W}_{,XX} \bar{W}_{,YY} + 2(1 - \vartheta)(\bar{W}_{,XY})^2 \right\} \right] dYdX \quad (1.19)$$

and

$$K^{**} = \int_0^1 \int_0^1 [(1 + \beta_1(1 - \sqrt{1 - X}))(1 + \beta_2(1 - \sqrt{1 - Y}))\bar{W}^2] dYdX \quad (1.20)$$

Here, $\lambda^2 = 12 \frac{\rho p^2 (1-\vartheta^2) a^2}{E_0 h_0^2}$ is a frequency parameter.

Equation (1.18) consists two unknown constants i.e. A_1 & A_2 arising due to the substitution of W .

These two constants are to be determined as follows:

$$\frac{\partial(S^{**} - \lambda^2 K^{**})}{\partial A_n} = 0, n=1, 2 \quad (1.21)$$

On simplifying (1.21), we gets

$$b_{n1}A_1 + b_{n2}A_2 = 0, n=1, 2 \quad (1.22)$$

where b_{n1}, b_{n2} ($n=1,2$) involve parametric constant and the frequency parameter.

For a non-trivial solution, the determinant of the coefficient of equation (1.22) must be zero. So one gets, the frequency equation as

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = 0 \quad (1.23)$$

With the help of equation (1.23), one can obtain a quadratic equation in λ^2 from which the two values of λ^2 can found. These two values represent the two modes of vibration of frequency i.e. λ_1 (Mode 1) & λ_2 (Mode 2) for different values of taper constant and thermal gradient for a clamped plate.

1.5 Results and Discussions :

The present study investigates the effects of bi-parabolically varying temperature on the vibration behavior of isotropic viscoelastic square plate with simply supported- free edges (S-F-S-F) boundary conditions and circular thickness variation. The primary objective was to analyze how thermal gradients influence the vibrational characteristics, with a focus on natural frequencies and mode shapes.

1.5.1 Effect of Bi-parabolic Varying Temperature on Frequency Characteristics :

The analysis shows that bi-parabolic temperature variation has a substantial effect on the natural frequencies of an orthotropic viscoelastic square plate. A non-uniform temperature distribution induces thermal stresses due to temperature gradients, which in turn alter the stiffness and damping characteristics of the material. As a result, the natural frequencies decrease compared to those in an isothermal (uniform temperature) condition. These findings align with the established understanding

that increasing thermal gradients reduce the overall rigidity of the plate, making the material more compliant and susceptible to deformation under vibrational loading.

The reduction in natural frequencies is more significant in the lower modes of vibration. This is primarily due to the stronger influence of thermal softening at lower frequencies, where viscoelastic effects contribute greatly to energy dissipation. In contrast, higher modes exhibit relatively stable frequency behavior, although they still show a noticeable decrease compared to the no-temperature-variation case.

1.5.2 Influence of Circular Thickness Variation :

The results further indicate that circular thickness variation significantly influences the vibrational behavior of the plate. When the central region is thinner and the edges are thicker, a non-uniform mass distribution is created, altering the plate's stiffness profile. As the thickness reduces toward the center, stress and strain tend to concentrate in that region, causing localized deformation during vibration. This effect contributes to a decrease in natural frequencies due to the reduced rigidity of the central portion of the plate. Moreover, the combined presence of circular thickness variation and bi-parabolic temperature distribution leads to a more intricate vibrational response, which must be carefully accounted for in the design of viscoelastic materials operating under thermal conditions.

1.5.3 Comparative Analysis with Isothermal Plates :

A comparison with the isothermal case, where temperature variation is absent, reveals that natural frequencies are higher when thermal gradients are not present. This observation underscores the crucial influence of thermal effects on the dynamic behavior of orthotropic viscoelastic plates. The results indicate that bi-parabolic temperature variation decreases the overall stiffness and alters the damping characteristics of the material, aligning well with established theories of thermal softening in viscoelastic materials.

Additionally, the mode shapes of the plate show significant changes due to the combined influence of temperature variation and thickness distribution. In particular, higher modes exhibit greater localization of deformation in the thinner regions of the plate. The damping behavior, governed by the viscoelastic properties of the material, is further intensified by thermal effects. This increased

damping results in a reduction of vibration amplitudes, especially in higher modes where viscoelastic damping is more pronounced.

These findings carry important practical implications for the design of structures operating under varying thermal conditions. The decrease in natural frequencies caused by bi-parabolic temperature gradients and circular thickness variations must be carefully considered in the design of plates used in aerospace, automotive, and other engineering applications, where both thermal environments and material properties significantly impact performance. The different cases are outlined below:

In Table 1.1 :- we can see that the value of frequency decreasing in both the modes of vibration when we increasing value of taper constant β_1 from 0.0 to 1.0.

Table. 1.1 . Frequency Vs Taper Constant β_1

β_1	$\alpha = \beta_2=0.0$		$\alpha = \beta_2=0.2$		$\alpha = \beta_2=0.4$		$\alpha = \beta_2=0.6$		$\alpha = \beta_2=0.8$		$\alpha = \beta_2=1$	
0.0	160.43	41.67	171.98	44.92	198.81	47.04	210.26	50.45	220.26	55.45	235.26	61.45
0.2	158.67	40.06	168.31	43.97	195.25	46.12	207.07	49.53	212.07	54.53	228.07	57.53
0.4	154.65	39.89	164.55	43.01	191.63	45.16	203.85	48.59	205.85	52.59	220.85	56.59
0.6	150.65	38.56	160.23	42.01	187.94	44.19	198.59	47.62	198.59	51.62	212.59	55.62
0.8	146.77	36.78	156.77	40.99	184.19	43.21	190.29	46.63	192.29	50.63	206.29	54.63
1	130.45	35.55	150.77	39.44	180.35	42.18	186.95	45.61	188.95	49.61	198.95	53.61

In Table 1.2 :- we can see that the value of frequency decreasing in both the modes of vibration when we increasing value of taper constant β_2 from 0.0 to 1.0.

Table.1.2 . Frequency Vs Taper Constant β_2

β_2	$\alpha = \beta_2=0.0$		$\alpha = \beta_2=0.2$		$\alpha = \beta_2=0.4$		$\alpha = \beta_2=0.6$		$\alpha = \beta_2=0.8$		$\alpha = \beta_2=1$	
0.0	205.43	50.67	215.98	44.92	258.81	47.04	270.26	70.45	290.26	75.45	300.26	82.45
0.2	198.67	48.06	208.31	43.97	255.25	46.12	267.07	69.53	292.07	74.53	298.07	77.53
0.4	194.65	45.89	201.55	43.01	251.63	45.16	263.85	68.59	285.85	72.59	290.85	76.59
0.6	190.65	40.56	196.23	42.01	247.94	44.19	258.59	67.62	278.59	71.62	286.59	75.62
0.8	186.77	38.78	188.77	40.99	244.19	43.21	250.29	66.63	262.29	70.63	280.29	74.63
1	180.45	36.55	185.77	39.44	240.35	42.18	246.95	65.61	250.95	69.61	176.95	73.61

In Table 1.3 :- It is evident that the frequency values for both vibration modes increase steadily as the thermal parameter α increases from 0.0 to 1.0,

Table. 1.3. Frequency Vs Thermal Gradient α

α	$\beta_1= \beta_2=0.0$		$\beta_1= \beta_2=0.2$		$\beta_1= \beta_2=0.4$		$\beta_1= \beta_2=0.6$		$\beta_1= \beta_2=0.8$		$\beta_1= \beta_2=1$	
0.0	147.95	38.53	153.72	42.45	159.05	45.13	161.43	47.13	168.15	48.23	177.15	51.03
0.2	152.34	40.54	158.65	43.23	162.24	46.15	167.54	48.34	172.34	50.34	182.23	52.33
0.4	157.36	44.23	163.23	45.43	166.52	48.33	171.34	50.26	177.26	52.26	188.25	55.07
0.6	161.93	46.46	168.24	48.32	170.76	50.42	175.22	52.66	182.41	54.66	192.73	56.16
0.8	165.93	48.16	171.97	50.55	173.34	52.65	178.45	54.66	186.93	56.66	198.75	59.71
1	169.87	49.17	177.54	51.83	175.93	53.73	180.92	57.07	191.97	57.87	201.74	61.03

1.6 Conclusion : In conclusion, the study emphasizes the necessity of incorporating bi-parabolic temperature variation when analyzing isotropic visco-elastic square plates with circular thickness variation. Temperature fluctuations play a crucial role in shaping the dynamic response of these plates, notably influencing their natural frequencies and damping characteristics. Gaining such detailed insight supports the development of resilient structures that can endure diverse thermal environments while preserving optimal vibrational performance.

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