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ON A LANDSBERG SPACE WITH THREE DIMENSIONS

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1 - INTRODUCTION :-

L.Berwald gave the concept of Landsberg Space in his earlier paper [1] . A Landsberg space is characterized by $P_{hijk} = 0$ in terms of cartain connection , where P_{hijk} is the hv- curvature tensor . The character of the space with $P_{hijk} = 0$ has not yet been sufficiently explored inspite of efforts of several authors. The condition P_{hij} is equivalent to C_{hij10}[2],[3] where C_{hij} is the (h)hv- torsion tensor , such a space of two dimensions was first considered by G.Landsberg [4] , so that we call such a space of general dimension as a landsberg space following A.Moor[9].

The purpose of present paper is to contribute a little to the theory of Landsberg space in a three dimensional Finsler space. A brief introduction of three dimensional Finsler space has been given in a section 2 of the present paper and the main result are stated in theorem 3.1 and 4.4 . In the last section we have considered on from Finsler metric in a three dimensional landsberg space and find out the relation between v- connection vectors and main scalars.

2 – THREE DIMENSIONAL FINSLER SPACE :-

In this section we give the fundamental formulas of three dimensional Finsler Space . Matsumato [5] developed theory of 3 dimensional Finsler space with respect to the orthogonal frame $e_{\alpha>i}$,

$\leftarrow \alpha = 1 ,2, 3$ where

$e_{1>}{}^{i} = L^{-1}y^{i} = l^{i}$ $e_{2>I} = m_i = C^{-1}C_i$ where C is the length of the torsion C_i . The third vector $e_{3>i}$ is given by $e_{3>i} = n_i = e_{ijk}\ e_{1>}{}^{j}\ e_{2>}{}^{k}$.

The (h) hv torsion tensor C_{ijk} are written as

(2.1) $LC_{ijk} = C_{\alpha\beta\Upsilon}\ e_{\alpha>i} e_{\beta>j}\ e_{\Upsilon>k}$

(2.2) $C_{1\alpha\beta} = 0$

$C_{222} = H$ $C_{223} = I$

and $C_{333} = -C_{233} = J$

which imply

(2.3) H +I = LC

The v-covariant derivative of the tensor $e_{\alpha>}{}^{i}$ are given by

(2.4) $e^{i}{}_{\alpha>Ij}L = \delta_{0\alpha}\,\delta_j{}^{i} - e_{\alpha>j}{}^{i} + \delta^{123}{}_{1\alpha\beta}\, e_{\beta>}{}^{i}\, V_j$

Where V_j are the components of the v – connenction vector .

The scalar component $T_{\alpha\beta\Upsilon}$ of a tensor $T_{jk}{}^{i}$ are defined by

(2.5) $T_{\alpha\beta\Upsilon} = T_{jk}{}^{i}\, e_{\alpha>i}\, e_{\beta>}{}^{j}\, e_{\Upsilon>}{}^{k}$

The scalar component $T_{\alpha\beta:\Upsilon}$ of $T^{i}{}_{j|k}L$ are written in the form

(2.6) $T_{\alpha\beta i\Upsilon} = L.d/dyk.T_{\alpha\beta}\, e_{\Upsilon>}{}^{k} + T_{\mu\beta}V_{\mu>\alpha\Upsilon} + T_{\alpha\mu}\, V_{\mu>\beta\Upsilon}$

Where $V_{\alpha>\beta\Upsilon}$ are such that

(2.7) $V_{\alpha>\beta\Upsilon} = -V_{\beta>\alpha\Upsilon}$, $V_{1>B\Upsilon} = \delta_{\beta\Upsilon} - \delta_{1\beta}\delta_{1\Upsilon}$ and $V_{2>3\Upsilon} = V_{\Upsilon}$

Thus the v- covariant derivatives of $T_j{}^{i}$ are given by

(2.8) $T^{i}{}_{jlk}L = T_{\alpha\beta:\Upsilon}\, e_{\alpha>}{}^{i}\, e_{\beta>j}\, e_{\Upsilon>k}$

Since $y_i = LdL/dy^{i}L$, in virtue of (2.1) and (2.8) we obtain

(2.9) $L^2C_{hijlk} + LC_{hij}\, e_{1>k} = C_{\alpha\beta\Upsilon:\delta}\, e_{\alpha>h}\, e_{\beta>I}\, e_{\Upsilon>j}\, e_{\delta>k}$

From (2.2) , (2.4) , (2.6) , and (2.7) we find

$$\begin{cases} C_{1\beta\Upsilon;\delta} = -C_{\beta\Upsilon\delta} \\ C_{222;\delta} = H_{;\delta} + 3JV_{\delta} \\ C_{223;\delta} = -J_{;\delta} + (H-2I)V_{\delta} \end{cases}$$

$C_{233;\delta} = I_{;\delta} + 3JV_{\delta}$

$C_{333;\delta} = J_{;\delta} + 3IV_{\delta}$

Where , $H_{;\delta} = L \cdot \partial H/\partial y^i \cdot e^i_{\delta>}$ similarly I;δ and J;δ

The v-connection vectors also satisfy the following relations :-

(2.11)

(a) $(H - 2I)V_2 - 3JV_3 = J_{;2} + H_{;3}$

(b) $3JV_2 + (H - 2I)V_3 = I_{;2} + J_{;3}$

(c) $3IV_2 + 3JV_3 = -J_{;2} + I_{;3}$

3 THE h -CURVATURE TENSOR OF LANDSBERG SPACE –

It is well known that the h-curvature tensor R_{hijk} are expressed as

(3.1) $R_{hijk} = R^{ab} e_{ahi} e_{bjk}$ where we put

(3.2) $R^{ab} = ¼.R_{hijk}\, e^{hia} e^{jkb}$ and e-tensor $e_{ijk} = \sqrt{g}\,.\delta^{123}_{ijk}$

Let us denote by $R_{\alpha\beta}$ the adapted components of the tensor R^{ab} then we have

(3.3) $R^{ab} = R_{\alpha\beta}\, e^a_{\alpha>} e^b_{\beta>}$

If $R_{\alpha\beta\Upsilon\delta}$ be the components of R_{hijk} , it is equivalent that

(3.4) $R_{hijk} = R_{\alpha\beta\Upsilon\delta}\, e_{\alpha>h}\, e_{\beta>i}\, e_{\Upsilon>j}\, e_{\delta>k}$

It follows from (3.1) that

(3.5) $R_{\alpha\beta\Upsilon\delta} = R_{\sigma\mu}\varepsilon_{\sigma\alpha\beta}\varepsilon_{\mu\Upsilon\delta}$

If we denote by $R_{\beta\Upsilon\delta}$ the adapted components of the (v)h-torsion tensor LR_{ijk} , then from (2.5) we have

(3.6) $R_{\beta\Upsilon\delta} = R_{\sigma\mu}\varepsilon_{\sigma1\beta}\varepsilon_{\mu\Upsilon\delta}$

because $L^{-1}R_{ijk} = e_{1>}{}^h R_{hijk}$

since the adapted components of $R_{jk|i}{}^{h}.L$ is $R_{\alpha\Upsilon\delta;\beta}$

therefore we have

(3.7) $R^{h}{}_{jk|I}.L = R_{\alpha\Upsilon\delta;\beta}\, e_{\alpha>}{}^{h}\, e_{\beta>i} e_{\Upsilon>j}\, e_{\delta>k}$

One of the Bianchi identities with respect to Cartain connection is

(3.8) $\mathfrak{S}_{(j,k)}\{R_{j}{}^{h}{}_{t}C_{ki}{}^{t} + P_{j}{}^{h}{}_{t}\, P_{k}{}^{t}{}_{i} + P^{h}{}_{ki|j}\} + R^{h}{}_{jk|i} - R^{h}{}_{ijk} = 0$

Where $\mathfrak{S}_{(jk)}$ {…} stands for interchanging in j and k and substracting .

Now we assume that F^3 is a three dimensional Landsberg space , then we have

(3.9) $P_{hijk} = 0$

Which implies $P_{0ijk} = P_{ijk} = 0$

Therefore Bianchi identity (3.8) reduces to

(3.10) $\mathfrak{S}_{(jk)}\{R_{jt}{}^{h}C_{ki}{}^{t}\} + R^{h}{}_{jk|i} - R^{h}{}_{ijk} = 0$

From the equation (2.1) and (3.7) , the above identity reduces to

$[\mathfrak{S}_{(\Upsilon\delta)}\{R_{\alpha\Upsilon\Phi}C_{\Phi\beta\delta}\} + R_{\alpha\Upsilon\delta;\beta} + R_{\alpha\Upsilon\delta}\,\delta_{1\beta} - R_{\beta\alpha\Upsilon\delta}]e_{\alpha>}{}^{h}\, e_{\beta>I}\, e_{\Upsilon>j}\, e_{\delta>k} = 0$

Which in view of (3.5) and (3.6) written as

(3.11) $\mathfrak{S}_{(\Upsilon\delta)}\{R_{\sigma\mu}\varepsilon_{\sigma1\alpha}\, \varepsilon_{\mu\Upsilon\Phi}\, C_{\Phi\beta\delta}\} + R_{\sigma\mu;\beta}\, \varepsilon_{\sigma1\alpha}\, \varepsilon_{\mu\Upsilon\delta} + R_{\sigma\mu}\varepsilon_{\sigma1\alpha}\varepsilon_{\mu\Upsilon\delta}$

$\delta_{0\beta} - R_{\sigma\mu}\varepsilon_{\sigma\beta\alpha}\varepsilon_{\mu\Upsilon\delta} = 0$

The identity (3.11) is equivalent to

$$(3.12)\quad \begin{cases} R_{11} = R_{12} = R_{13} = 0 \\ R_{21} = R_{22} = R_{23} = 0 \\ R_{31} = R_{32} = R_{33} = 0 \end{cases}$$

In view of (3.4) and,(3.5) and (3.12) we have

$R_{hijk} = 0$

Consquently we have an important theorem.

THEOREM 3 :- Let F^3 be a three dimensional non Riemannian Landsberg space , then F^3 is flat i.e h-curvature tensor $R_{hijk} = 0$.

4. LANDSBERG SPACE WITH 1-FORM METRIC –

We shall consider a three dimensional Finsler space F^3 with the fundamental fuctional $L = L(a^\alpha)$([6],[7]), $a^\alpha = a^\alpha{}_i(x)y^i$, where $L(a^\alpha)$ is (1)p-homogeneous in $a^\alpha(\alpha=1,2,3)$ and $a^\alpha{}_i(x)\, dx^i$ are linearly independent differential 1-forms , that is a det.$(a_i{}^\alpha) \neq 0$. The Finsler metric $L = L(a^\alpha)$, $a^\alpha = a^\alpha{}_i(x)y^i$ is called a 1 form Finsler metric and the Finsler space F^3 is a 1-form Finsler space in three dimensional Finsler space.

We denote C1 = $(\Gamma^i{}_{jk} , \Gamma^i{}_{oj} ,C^i{}_{jk})$ the 1- form cartan connection [6].

Let $C\Gamma = (F^i{}_{jk} ,F^i{}_{oj} , C^i{}_{jk})$ be the Cartan connection of the F^3 , then the difference tensor $D^i{}_{jk} = F^i{}_{jk} - \Gamma^i{}_{jk}$ of $C\Gamma$ from C1 [6] is given by

(4.1) $D_{ijk} = -A_{ijk} + C_i{}^\Upsilon{}_j.A_{\Upsilon k} + C_j{}^\Upsilon{}_k.A_{\Upsilon i} - C_k{}^\Upsilon{}_i.A_{\Upsilon j}$

Where we put

(4.2) (a) $A_{jk} = A_{ojk} - C_j{}^\Upsilon{}_k A_{o\Upsilon o}$

where $A_{ojk} = A_{hjk}y^h$ i.e o denotes the contraction by y^i

(b) $2A_{ijk} = T_{ijk} - T_{jki} + T_{kij}$

And $T_{ijk} = g_{\Upsilon j}.T_{ik}{}^\Upsilon$ $T_j{}^i{}_k$ is (h)h- torsion tensor.

A Finsler space F^n with 1 form metric is a Landsberg space if and only if [8] .

(4.3) $C_i{}^\Upsilon{}_{j/h}A_{ro} + \mathfrak{S}(hij) \{C_i{}^\Upsilon{}_j.A_{h\Upsilon 0}\} = 0$

Where $\mathfrak{S}(h_{ij})\{\ldots\}$ stands for cyclic permutation h,I,j and summation.

We shall restrict our discussion to a three dimensional Finsler space F^3 with 1-form metric.

We denote $T_{\beta\Upsilon\delta}$, the adopted components of T_{ijk} ,then we have

(4.4) $T_{ijk} = T_{\beta\Upsilon\delta} e_{\beta>i} e_{\Upsilon>j} e_{\delta>k}$

Then the euation (4.2) (b) may be written as

(4.5) $A_{ijk} = ½L(T_{\beta\Upsilon\delta} - T_{\Upsilon\delta\beta} + T_{\delta\beta\Upsilon}) e_{\beta>i} e_{\Upsilon>j} e_{\delta>k}$

The contraction with y^i equation (4.5) yields

(4.6) $A_{ojk} = 1/2(L)(T_{1\Upsilon\delta} - T_{\Upsilon\delta 1} + T_{\delta 1\Upsilon}) e_{\Upsilon>j} e_{\delta>k}$

Again contraction with y^k , above equation gives

(4.7) $A_{ojo} = 1/2L^2 T_{11\Upsilon} e_{\Upsilon>j}$

Substuting (2.1),(4.6),(4.7) in the equation (4.2)(a) we obtain

(4.8) $A_{jk} = \{1/2L(T_{1\Upsilon\delta} - T_{\Upsilon\delta 1} + T_{\delta 1\Upsilon}) - C_{\Upsilon\delta o} L T_{11\Upsilon}\} e_{\Upsilon>j} e_{\delta>k}$

contracting the equation (4.5) and (4.8) by y^k respectively we have

(4.9) $A_{ijo} = 1/2(T_{\beta\Upsilon 1} - T_{\Upsilon 1\beta} + T_{1\beta\Upsilon}) e_{\beta>i} e_{\Upsilon>j}$

(4.10) $A_{jo} = L^2 T_{11\Upsilon} e_{\Upsilon>j}$

Let $C_{\beta\Upsilon\Phi;\alpha}$ be the adapted component of $LC_{ij}{}^{\Upsilon}{}_{/h}$,then we have as (2.9) .

(4.11) $C_i{}^{\Upsilon}{}_{j|h} = L^{-2}(C_{\beta\Upsilon\Phi;\alpha} - C_{\beta\Upsilon\Phi}\delta_{1\alpha}) e_{\alpha>h} e_{\beta>i}\, e_{\Upsilon>j}\, e_{\Phi>}{}^{\Upsilon}$

From (2.1),(4.9),(4.10) equation (4.3) gives

(4.12) $[(C_{\beta\Upsilon\Phi;\alpha} - C_{\beta\Upsilon\Phi}\delta_{1\alpha}) T_{11\Phi} + Ф_{(\alpha\beta\Upsilon)}\{1/2C_{\beta\Upsilon\Phi}(T_{\alpha\Phi 1} - T_{\Phi 1\alpha} + T_{1\alpha\Phi})\}] e_{\alpha>h} e_{\beta>i} e_{\Upsilon>j} = 0$

In view of (2.2) and(2.10), the equation (4.12) may be written as

(4.13) (i) $(H_{;2} + 3JV_2)T_{112} + \{-J_{i2} + (H-2I)V_2\}.T_{113} - 3JT' = 0$

(ii) $\{-J_{i2} + (H-2I)V_2\}T_{112} + (I_{;2} - 3JV_2)T_{113} – (H-2I)T' = 0$

(iii) $(I_{;2} – 3JV_2)T_{112} + (J_{;2} + 3IV_2)T_{113} + 3JT' = 0$

(iv) $(H_{;3} + 3JV_3)T_{112} + \{-J_{;3} + (H-2I)V_3\}T_{113} - (H-2I)T' = 0$

(v) $\{-J_{;3} + (H-2I)V_3\}T_{112} + (I_{;3} - 3JV_3)T_{113} + 3JT' = 0$

(vi) $(I_{;3} - 3JV_3)T_{112} + (J_{;3}+3IV_3)T_{113} - 3IT' = 0$

Where we have put $T' = 1/2(T_{231} - T_{312} + T_{123})$

In view of(2.11), it follows that equations (ii) and (iv) equation (iii) and (v) of (4.13) are equivalent . Hence there are only four independent equations in three unknowns T_{112} , T_{113} , T^1 .

Further more , the addition of (4.13)(ii) and (4.13)(vi) , and use of (2.11)(c) give the equation .

(4.14) $V_2T_{112} + V_3T_{113} - T^1 = 0$

Adding (4.13)(i) and (4.13)(iii) and using the equation (2.3) we obtain

(4.15) $C_{;2}T_{112} + CV_2T_{113} = 0$

If the scalar components of $T_i{}^h{}_j$ has non vanishing components T_{112} , T_{113} and T' , then eliminating them from (4.13)(vi) ,(4.14) and (4.15) , we get

$$\begin{vmatrix} I_{;3} - 3JV_3 & J_{;3} + 3IV_3 & -3I \\ V_2 & V_3 & -1 \\ C_{;2} & CV_2 & 0 \end{vmatrix} = 0$$

<u>THEOREM 4.1 -</u> A three dimensional Finsler space F^3 with 1-form metric is a Landsberg space if and only if T_{112} , T_{113} and T^1 vanishes.

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