On Absolute Matrix Summability Factors using Quasi β-Power Increasing Sequences

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Abstract:

In this note two theorems concerning the absolute matrix summability factors have been established by using quasi β-power increasing sequences. These theorems generalize some known results and also give rise to some new factor theorems.

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1 Introduction

A positive sequence $\{g_n\}$ is said to be quasi β-power increasing sequence if there exists a constant $M = M(\beta, g_n)$ such that $M\, n^\beta g_n \geq m^\beta g_m$ holds for all $n \geq m \geq 1$ (see [4]).

Let $A = (a_{nm})$ be a lower triangular matrix of nonzero diagonal entries (we call such a matrix a normal matrix). Then A defines a sequence to sequence transformation mapping the sequence $s = \{s_n\}$ to $A_s = \{A_n(s)\}$, where

$$A_n(s) = \sum_{k=0}^{n} a_{nm}\, s_m \tag{1.1}$$

The series $\sum a_n$ is said to be summable $|A|_k$, $k \geq 1$, if (see [9])

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta A_n(s)|^k < \infty \tag{1.2}$$

where

$$\Delta A_n(s) = A_n(s) - A_{n-1}(s)$$

For a sequence $\{p_n\}$of positive numbers, we write

$$P_n = \sum_{m=1}^{n} p_m \to \infty \;\; as \;\; n \to \infty, \;\; (P_{-i} = p_{-i} = 0, i \geq 1 \tag{1.3}$$

The series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geq 1$ if (see [8])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \, |\Delta A_n(s)|^k < \infty \tag{1.4}$$

The series $\sum a_n$ is said to be summable $|A, \delta\,|_k$, if

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}|\Delta A_n(s)|^k < \infty \tag{1.5}$$

The series $\sum a_n$ is said to be summable $|A, p_n, \delta|_k$, if (see [5])

$$\sum_{n=1}^{\infty}(P_n/p_n)^{\delta k+k-1} |\Delta A_n(s)|^k < \infty \tag{1.6}$$

When $p_n = 1$, the summability $|A, p_n, \delta|_k$ reduces to $|A, \delta|_k$ summability. For $\delta = 0$, the summability $|A, p_n, \delta|_k$ is same as $|A, p_n|_k$ summability. Finally when $a_{nk} = p_k/P_n$, the summability $|A, p_n, \delta|_k$ reduces to the method $|\bar{N}, p_n, \delta|_k$ method.

We shall use the two lower submatrices $\bar{A} = (\bar{a}_{nm})$ and $\hat{A} = (\hat{a}_{nm})$ associated with the normal matrix $A = (a_{nm})$, which we define as follows:

$$\bar{a}_{nm} = \sum_{j=m}^{n} a_{nj} \tag{1.7}$$

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nm} = \bar{a}_{nm} - \bar{a}_{n-1,m} \quad n = 1, 2, \ldots \tag{1.8}$$

It is well-known that

$$A_n(s) = \sum_{m=0}^{n} a_{nm} s_m = \sum_{m=0}^{n} \bar{a}_{nm} a_m \tag{1.9}$$

and

$$\Delta A_n(s) = \sum_{m=0}^{n} \hat{a}_{nm} a_m \tag{1.10}$$

2 Some known results

Recently the concept of quasi β-power increasing sequence has been utilized by many researchers to obtain summability factor theorem (see, e.g., [4], [2], [3] and [1]).

The following two theorems were proved for $|A, p_n|_k$ summability.

Theorem 2.1 [3]: Let $A = (a_{nm})$ be a positive normal matrix such that

$$\bar{a}_{n0} = 1, \quad n = 0, 1, \ldots \tag{2.1.1}$$

$$a_{n-1,m} \geq a_{nm}, \quad for\ n \geq m+1 \tag{2.1.2}$$

$$a_{nn} = O(p_n/P_n) \tag{2.1.3}$$

and $\{X_n\}$ be a quasi β-power increasing sequence for some $0 < \beta < 1$. Let $\{b_n\}$ and $\{l_n\}$ be sequences such that

$$|\Delta l_n| \leq b_n \tag{2.1.4}$$

$$b_n \to 0 \ \ as \ \ n \to \infty \tag{2.1.5}$$

$$\sum_{n=1}^{\infty} n|\Delta b_n|X_n < \infty, \tag{2.1.6}$$

$$|l_n|\, X_n = O(1), \ \ as\, n \to \infty \tag{2.1.7}$$

If

$$\sum_{m=1}^{n} \frac{|s_m|^k}{m} = O(X_n), \tag{2.1.8}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n}|s_n|^k = O(X_m), \ \ m \to \infty \tag{2.1.9}$$

then $\sum a_n l_n$ is summable $|\bar{N}, \ p_n|_k$, $k \geq 1$.

Theorem 2.2 [3]. Let $\{X_n\}$ be a quasi β-power increasing sequence for some $0 < \beta < 1$. Let $\{b_n\}$ and $\{l_n\}$satisfy the conditions (2.1.4) - (2.1.7) and (2.1.9). If

$$\sum_{n=1}^{\infty} P_n|\Delta b_n|X_n < \infty \tag{2.2.1}$$

$$\sum_{n=1}^{m} \frac{|s_n|^k}{P_n} = O(X_m), \ \ m \to \infty \tag{2.2.2}$$

then $\sum a_n l_n$ is summable $|\bar{N}, \ p_n|_k$, $k \geq 1$.

The above two theorems were generalized for $|A, p_n|_k$ summability as follows:

Therem 2.3 [6]. Let $A = (a_n)$ be a positive normal matrix such that

$$\bar{a}_{n0} = 1, \ \ \ n = 0, 1, 2. \cdots \tag{2.3.1}$$

$$a_{n-1,m} \geq a_{nm} \ \ \ for\, n \geq m+1 \tag{2.3.2}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{2.3.3}$$

and if $\{X_n\}$ is a quasi β-power increasing sequence for some $0 < \ \beta < 1$ and the sequences $\{b_n\}$ and $\{l_n\}$ satisfy the conditions (2.1.1) – (2.1.9) and

$$\{l_n\} \in \mathcal{BV} \tag{2.3.4}$$

is satisfied then $\sum a_n l_n$ is summable $|A, \ p_n|_k$, $k \geq 1$.

In the special case when $a_{nm} = p_m / P_n$, this theorem reduces to Theorem 2.1.

Theorem 2.4 [6]. Let $A = (a_{nm})$ be a positive normal matrix as in Theorem 2.3 and let $\{X_n\}$ be a quasi β-power increasing sequence for some $0 < \ \beta < 1$. If all the conditions of Theorem 2.2 and (2.3.4) are satisfied then the series $\sum a_n l_n$ is summable $|A, \ p_n|_k$, $k \geq 1$.

The above two theorems (Theorems 2.3 and 2.4) were extended for $|A, p_n, \delta|_k$ summability in the following form.

Theorem 2.5 [7]. Let $A = (a_{nm})$ be a positive normal matrix such that

$$\bar{a}_{n0} = 1, \quad n = 0, 1, 2. \cdots \tag{2.5.1}$$

$$a_{n-1,m} \geq a_{nm} \quad for\, n \geq m+1 \tag{2.5.2}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{2.5.3}$$

$$\sum_{n=m+1}^{j+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_m \hat{a}_{nm}| = O\left\{\left(\frac{P_m}{P_m}\right)^{\delta k-1}\right\} \tag{2.5.4}$$

$$\sum_{n=m+1}^{j+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_m \hat{a}_{n,m+1}| = O\left\{\left(\frac{P_m}{P_m}\right)^{\delta k}\right\} \tag{2.5.5}$$

and let there be sequences $\{b_n\}$ and $\{l_n\}$such that

$$\{l_n\} \in \mathcal{BU}, \tag{2.5.6}$$

$$|\Delta l_n| \leq b_n, \tag{2.5.7}$$

$$b_n \to 0 \quad as \quad n \to \infty \tag{2.5.8}$$

$$\sum_{n=1}^{\infty} n|\Delta b_n| X_n < \infty, \tag{2.5.9}$$

$$|l_n|\, X_n = O(1) \quad n \to \infty \tag{2.5.10}$$

where $\{X_n\}$ is a quasi β-power increasing sequence for some $0 < \beta < 1$. If

$$\sum_{m=1}^{n} \left(\frac{P_m}{p_m}\right)^{\delta k} \frac{|s_m|^k}{m} = O(X_n) \tag{2.5.11}$$

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k-1} |s_n|^k = O(X_m), \; m \to \infty \tag{2.5.12}$$

then $\sum a_n l_n$ is summable $|A, p_n, \delta|_k$, $k \geq 1$ and $0 \leq \delta < \frac{1}{k}$.

Theorem 2.6 [7]. Let conditions (2.5.1) – (2.5.10) and (2.5.12) of Theorem 2.5 be satisfied. If

$$\sum_{n=1}^{\infty} P_n |\Delta b_n| X_n < \infty, \tag{2.6.1}$$

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k} \frac{|s_n|^k}{P_n} = O(X_m) \tag{2.6.2}$$

then $\sum a_n l_n$ is summable $|A, p_n, \delta|_k$, $k \geq 1$ and $0 \leq \delta < \frac{1}{k}$.

3 Main Results

Before we give the generalizations of Theorems 2.5 and 2.6, we introduce the following:

Definition 3.1. Let $A = (a_{nm})$ be a positive normal matrix and $\{\varphi_n\}$ be a sequence of positive numbers. Then the infinite series $\sum a_n$ is said to be summable $|A, \varphi_n, \delta|_k, \delta \geq 0, k \geq 1$ if

$$\sum_{n=1}^{\infty}(\varphi_n)^{\delta k+k-1}\,[\Delta A_n(s)]^k < \infty.$$

Theorem 3.1: Let $A = (a_{nm})$ be a positive normal matrix such that

$$\bar{a}_{n0} = 1, \quad n = 0, 1, 2. \cdots \tag{3.1.1}$$

$$a_{n-1,m} \geq a_{nm} \quad for\; n \geq m+1 \tag{3.1.2}$$

$$a_{nn} = O(\varphi_n), \tag{3.1.3}$$

$$\sum_{n=m+1}^{j+1} \varphi_n^{\delta k} |\Delta_m \hat{a}_{nm}| = O\{\varphi_m^{\delta k-1}\} \tag{3.1.4}$$

$$\sum_{n=m+1}^{j+1} \varphi_n^{\delta k} \left|\Delta_m \hat{a}_{n,m+1}\right| = O\{\varphi_m^{\delta k}\} \tag{3.1.5}$$

and let there be sequences $\{b_n\}$ and $\{l_n\}$such that

$$\{l_n\} \in \mathcal{BV}, \tag{3.1.6}$$

$$|\Delta l_n| \leq b_n, \tag{3.1.7}$$

$$b_n \to 0 \quad as \quad n \to \infty \tag{3.1.8}$$

$$\sum_{n=1}^{\infty} n|\Delta b_n| X_n < \infty, \tag{3.1.9}$$

$$|l_n|\, X_n = O(1) \quad n \to \infty \tag{3.1.10}$$

where $\{X_n\}$ is a quasi β-power increasing sequence for some $0 < \beta < 1$. If

$$\sum_{m=1}^{n} \varphi_m^{\delta k} \frac{|s_m|^k}{m} = O(X_n) \tag{3.1.11}$$

$$\sum_{n=1}^{m} \varphi_n^{\delta k-1} |s_n|^k = O(X_m), \; m \to \infty \tag{3.1.12}$$

then $\sum a_n l_n$ is summable $|A, \varphi_n, \delta|_k$, $k \geq 1$ and $0 \leq \delta < \frac{1}{k}$.

Theorem 3.2: Let conditions (3.1.1) – (3.1.10) and (3.1.12) of Theorem 3.1 be satisfied. If

$$\sum_{n=1}^{\infty} \varphi_n |\Delta b_n| X_n < \infty, \tag{3.2.1}$$

$$\sum_{n=1}^{m} \varphi_n^{\delta k-1} |s_n|^k = O(X_m) \tag{3.2.2}$$

then $\sum a_n l_n$ is summable $|A, \varphi_n, \delta|_k$, $k \geq 1$ and $0 \leq \delta < \frac{1}{k}$.

4 Some Lemmas

We shall need the following lemmas for the proof of our theorems.

Lemma 4.1 (Leindler, 2001): Let the sequence $\{X_n\}$ be quasi β – power increasing for some $0 < \beta < 1$. If conditions (3.1.8) and (3.1.9) are satisfied, then

$$nX_n b_n = O(1) \quad as \quad n \to \infty \tag{4.1.1}$$

$$\sum_{n=1}^{\infty} X_n b_n < \infty \tag{4.1.2}$$

Lemma 4.2: Let $\{X_n\}$ be a quasi β–power increasing sequence for $0 < \beta < 1$. Let φ_n be a sequence such that $\{\varphi_n X_n\}$ is increasing. If conditions (3.1.8) and (3.2.1) are satisfied, and

$$\sum_{n=1}^{j} \varphi_n X_n = O(\varphi_j X_j),$$

then,

$$\varphi_n b_n X_n = o(1) \tag{4.2.1}$$

$$\sum_{n=1}^{\infty} \varphi_n b_n X_n < \infty. \tag{4.2.2}$$

Proof. Since $b_n \to 0 \;\; as \;\; n \to \infty$, we have

$$b_n = \sum_{j=n}^{\infty} \Delta b_j . \tag{4.2.3}$$

Since $\{\varphi_n X_n\}$ is increasing, we have

$$\varphi_n b_n X_n \leq \sum_{j=n}^{\infty} \varphi_j |\Delta b_j| X_j < \infty$$

By (3.2.1). Hence

$$\varphi_n b_n X_n = o(1), \;\; as \;\; n \to \infty.$$

Further, using (4.2.3) and the fact that $\{X_n\}$ is increasing, we have

$$\sum_{n=1}^{\infty} \varphi_n b_n X_n = \sum_{n=1}^{\infty} \varphi_n X_n \sum_{j=n}^{\infty} |\Delta b_j| \leq \sum_{j=1}^{\infty} |\Delta b_j| \sum_{n=1}^{j} \varphi_n X_n$$
$$\leq M \sum_{j=1}^{\infty} |\Delta b_j| X_j \varphi_j < \infty$$

This completes the proof of the lemma.

5 Proofs of the theorems

5.1 Proof of Theorem 3.1

Let $\{Y_n\}$be the A-transform of the series $\sum a_n l_n$. Now invoking Abel's transformation on equations (1.9) and (1.10), we obtain

$$\begin{aligned}
\Delta Y_n &= \sum_{m=1}^{n} \hat{a}_{nm} a_m l_m \\
&= \sum_{m=1}^{n-1} \Delta_m (\hat{a}_{nm} l_m) \sum_{j=1}^{m} a_j + \hat{a}_{nn} \sum_{m=1}^{n} a_m \\
&= \sum_{m=1}^{n-1} (\hat{a}_{nm} l_m - \hat{a}_{n,m+1} l_{m+1}) s_m + a_{nn} l_n s_n \\
&= \sum_{m=1}^{n-1} (\hat{a}_{nm} l_m - \hat{a}_{n,m+1} l_{m+1} - \hat{a}_{n,m+1} l_m + \hat{a}_{n,m+1} l_m) s_m + a_{nn} l_n s_n \\
&= \sum_{m=1}^{n-1} \Delta_m (\hat{a}_{nm}) l_m s_m + \sum_{m=1}^{n-1} \hat{a}_{n,m+1} \Delta l_m s_m + a_{nn} l_l s_n \\
&= Y_{n1} + Y_{n2} + Y_{n3}, \quad say.
\end{aligned} \tag{5.1.1}$$

Now, since

$$|a + b + c|^k \leq 3^k (|a|^k + |b|^k + |c|^k),$$

in order to complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k + k - 1} |Y_{ni}|^k < \infty, \quad for \; i = 1, 2, 3. \tag{5.1.2}$$

Now we shall apply Holder's inequality with indices k and k', where k > 1 and $\frac{1}{k}+\frac{1}{k'}=1$, we obtain

$$\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}|Y_{n1}|^k \le \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left(\sum_{j=1}^{n-1}|\Delta_j\hat{a}_{nj}||l_j||s_j|\right)^k$$

$$= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left(\sum_{j=1}^{n-1}|\Delta_j\hat{a}_{nj}||l_j|^k|s_j|^k\right)\times\left(\sum_{j=1}^{n-1}|\Delta_j\hat{a}_{nj}|\right)^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k}\left(\sum_{j=1}^{n-1}|\Delta_j\hat{a}_{nj}||l_j|^k|s_j|^k\right)$$

$$= O(1)\sum_{j=1}^{m}|l_j|^k|s_j|^k\sum_{n=j+1}^{m+1}\varphi_n^{\delta k}|\Delta_j\hat{a}_{nj}|$$

$$= O(1)\sum_{j=1}^{m}\varphi_j^{\delta k-1}|l_j|^{k-1}|l_j||s_j|^k$$

$$= O(1)\sum_{j=1}^{m}\varphi_j^{\delta k-1}|l_j||s_j|^k$$

$$= O(1)\sum_{j=1}^{m-1}\Delta|l_j|\sum_{i=1}^{j}\varphi_i^{\delta k-1}|s_i|^k + O(1)|l_m|\sum_{j=1}^{m}\varphi_j^{\delta k-1}|s_j|^k$$

$$= O(1)\sum_{j=1}^{m-1}b_jX_j + O(1)|l_m|X_m$$

$$= O(1) \quad as \quad m\to\infty$$

by enforcing the hypotheses of Theorem 3.1 and Lemma 4.1.

Now using condition (3.1.6) and again applying Holder's inequality with the same indices, we obtain

$$\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}|Y_{n2}|^k \le \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left(\sum_{j=1}^{n-1}\hat{a}_{n,j+1}\Delta l_j s_j\right)^k 1$$

$$= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left(\sum_{j=1}^{n-1}|\hat{a}_{n,j+1}||\Delta l_j||s_j|^k\right)\times\left(\sum_{j=1}^{n-1}|\hat{a}_{n,j+1}||\Delta l_j|\right)^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k}\left(\sum_{j=1}^{n-1}b_j|\hat{a}_{n,j+1}||s_j|^k\right)\times\left(\sum_{j=1}^{n-1}|\Delta l_j|\right)^{k-1}$$

$$= O(1)\sum_{j=1}^{m}b_j|s_j|^k\sum_{n=j+1}^{m+1}\varphi_n^{\delta k}|\hat{a}_{n,j+1}|$$

$$= O(1)\sum_{j=1}^{m}\varphi_j^{\delta k}b_j|s_j|^k$$

$$= O(1)\sum_{j=1}^{m}\varphi_j^{\delta k}\frac{|s_j|^k}{j}(jb_j)$$

$$= O(1)\sum_{j=1}^{m-1}\Delta(jb_j)\sum_{i=1}^{j}\varphi_i^{\delta k}\frac{|s_i|^k}{i} + O(1)\left[mb_m\sum_{j=1}^{m}\varphi_j^{\delta k}\frac{|s_j|^k}{j}\right]$$

$$= O(1)\sum_{j=1}^{m-1}\Delta(jb_j)X_j + O(1)\{mb_mX_m\}$$

$$= O(1)\sum_{j=1}^{m-1}j|\Delta b_j|X_j + O(1)\sum_{j=1}^{m-1}b_{j+1}X_{j+1} + O(1)\{mb_mX_m\}$$

$$= O(1) \quad as \quad m\to\infty,$$

by utilising the hypotheses of Theorem 3.1 and Lemma 4.2.

Finally, using the same arguments as in the case of Y_{n1} , we obtain

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1}|Y_{n3}|^k \leq \sum_{n=1}^{m} \varphi_n^{\delta k+k-1}|a_{nn}|^k|l_n|^k|s_n|^k$$
$$= O(1)\sum_{n=1}^{m} \varphi_n^{\delta k-1}|l_n||s_n|^k$$
$$= O(1) \quad as \quad m \to \infty.$$

Consequently, we have shown that

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1}|Y_{ni}|^k < \infty, \qquad for \quad i = 1,2,3.$$

This completes the proof of Theorem 3.1.

5.2 Proof of Theorem 3.2

The proof of this theorem can be written proceeding in the same way as in Theorem 3.2, using Lemma 4.2 and substituting

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+1} b_n|s_n|^k$$

for

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k} b_n|s_n|^k.$$

6. Concluding Remarks

We have proved two theorems for the absolute summability factors for the method $|A,\varphi_n,\delta|_k$. We have suggested a more logical definition for this method, which generalizes the definition of $|A,p_n,\delta|_k$. By taking the sequence $\varphi_n = \frac{P_n}{p_n}$, we obtain Theorem 2.5 and a slightly modified version of Theorem 2.6. This, in turn, gives us all the other results cited herein by taking suitable values of the matrix $A = (a_{nm})$, the sequence φ_n and δ.

REFERENCES

[1] Bor, H. (2008), "On a new application of power increasing sequences, *Proc. Etonian Acad. Sci*., 57 (**4**), 205 – 209.

[2] Bor, H. and Debnath, L. (2004), Quasi power β–increasing sequences, Internat. *Jour. Math & Math. Sci*. 44, 2371 – 2376.

[3] Bor, H. and Ozarslan (2000), On the quasi power increasing sequences, *Jour. Math. Anal Appl.* 276, 924 – 929.

[4] Leindler, L. (2001), A new application of quasi power increasing sequences, *Publ. Math. Debrecen*, 58, 791 – 796.

[5] Őzarslan, H. S. and Őgduk, H. N. (2004), Generalization of two theorems on absolute summability methods, *Aust. J. Math. Anal Appl.* 1 (**2**), 1 – 7.

[6] Ȍzarslan, H. S. and Yavuz, E. (2013), A new note on absolute matrix summability, *Jour. Inequl. Appl*., 474, 1 – 7.

[7] Ȍzarslan, H. S. and Yavuz, E. (2014), New theorems for absolute matrix summability factors, *Gen. Math. Notes*, 27(**2**), 63 – 70.

[8] Sulaiman, W. T. (2003), Inclusion theorems for absolute matrix summability methods of infinite series (IV), *Indian J. Pure Appl. Math.* 34(**11**), 1547 – 1557.

[9] Tanović-Miller, N. (1979), On strong summability, *Glas. Math*. 34(**14**), 87 – 97.