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ON BOUNDS FOR LOGARITHMIC NON-SYMMETRIC WEIGHTED DIVERGENCE MEASURES

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ABSTRACT
In the present communication, we have classified the classical divergence measures into logarithmic, non-logarithmic, symmetric and non-symmetric categories. Here we have considered a sequence of bounds for logarithmic-non-symmetric weighted divergence measures.
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1.1. INTRODUCTION

Divergence measures have played a vital role to test reliability of the information, statement or the system. To minimize or maximize the same depends upon the goal or strategy of the experimenter, Recently Ruchi and Singh [3] has applied different divergence measures for profit maximization in share market. The same study has been extended to decision making process in case of world universities ranking problems. To correlate the different parameters for ranking problem, divergence measures have been tested.

Recently, Taneja [4] has made a lot of contribution to the studies of different types of divergence measures, specially symmetric and non-symmetric. Classical and some new divergence measures have been used for relationship among them in terms of inequalities.

Authors in one communication, have classified the classical divergence measures into logarithmic and non-logarithmic, symmetric and non-symmetric divergence measures. Convexity property has been features, exploring calculus and Csiszar's [1] f-divergence. Now in this paper, we consider bounds among logarithmic-non-symmetric weighted divergence measures in the next section.

SECTION 2

(a) Let us have different logarithmic-non-symmetric weighted divergence measures:

(i) $$LN_K(P\|Q;W)=\sum_{i=1}^{n} w_i p_i \log\frac{p_i}{q_i}, \tag{2.1}$$

(ii) $$LN_D(P\|Q;W)=\sum_{i=1}^{n} w_i (p_i-q_i)\log\left(\frac{p_i+q_i}{2q_i}\right) \tag{2.2}$$

(iii) $$LN_F(P\|Q;W)=\sum_{i=1}^{n} w_i p_i \log\left(\frac{2p_i}{p_i+q_i}\right) \tag{2.3}$$ and

(iv) $$LN_G(P\|Q;W)=\sum_{i=1}^{n} w_i \left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2p_i}\right) \tag{2.4}$$

(b) The second type of logarithmic weighted symmetric divergence are:

(i) $$LS_J(P\|Q;W)=\sum_{i=1}^{n} w_i(p_i-q_i)\log\frac{p_i}{q_i} \tag{2.5}$$

(ii) $$LS_I(P\|Q;W)=\frac{1}{2}\left[\sum_{i=1}^{n} w_i p_i \log\left(\frac{2p_i}{p_i+q_i}\right)+\sum_{i=1}^{n} w_i q_i \log\left(\frac{2q_i}{p_i+q_i}\right)\right] \tag{2.6}$$

and

(iii) $$LS_T(P\|Q;W)=\sum_{i=1}^{n} w_i\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2\sqrt{p_i q_i}}\right) \tag{2.7}$$

SECTION 3

CSISZAR'S F-DIVERGENCE AND TANEJA'S [2] EXTENSION

Let f : $(0, \infty) \rightarrow R$ be a convexed function, the f-divergence measure due to Csiszar[1] is given by

$$C_f(P\|Q)=\sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right) \tag{3.1}$$

where P, Q, $\in\Gamma_n$, and

$$\Gamma_n=\{P=(p_1,p_2,...,p_n): p_i>0,\ \sum_{i=-1}^{n} p_i=1\},\ n\geq 2\,. \tag{3.2}$$

- **Csiszar's Theorem:** Let the function f : $(0, \infty) \rightarrow R$ be differentiable convex and normalized, i.e. f(1) = 0, then the f-divergence (3.1)C_f (P || Q) is non-negative and convex in the pair of probability distribution (P, Q) $\in\Gamma_n\times\Gamma_r$.

Dragomir [2] extended (3.1) as:

- **Dragomir's Theorem:** If f : $R_+\rightarrow R$be a differentiable convex and normalized function i.e. f(1) = 0. Then

$$0\leq C_f(P\|Q)\leq E_{C_f}(P\|Q) \tag{3.3}$$

where

$$E_{C_f}(P\|Q)=\sum_{i=1}^{n}(p_i-q_i)f'\left(\frac{p_i}{q_i}\right) \tag{3.4}$$

$\forall$P, Q $\in\Gamma_n$.

- **Taneja's Theorem:** Taneja [4] extended Csiszar's f-divergence as follows:

Let f_1, f_2: I $\subset$ R $\rightarrow$R be two differentiable convex and normalized functions i.e. $f_1(1)=f_2(1)=0$ and

(i) f_1 and f_2 are twice differentiable in (r, R) : where $0 < r \leq p_i/q_i \leq 1 \leq R < \infty$,

(ii) then there exist the real constants m and M such that m < M and

$$m\leq\frac{f_1''(x)}{f_2''(x)}\leq M,\quad f_2''(x)>0,\quad \forall\ x\in(r,R) \tag{3.5}$$

Then $$mC_{f_2}(P\|Q)\leq C_{f_1}(P\|Q)\leq MC_{f_2}(P\|Q) \tag{3.6}$$

Ruchi and Singh [3] extended Taneja's theorem for weighted distribution.

$W=(w_1,w_2,...,w_n)$, $w_i > 0$, $\forall$ i, 2, ...,n,

corresponding to probability distribution and extended the result (3.6) as.

$$mC_{f_2}(P\|Q;W)\leq C_{f_1}(P\|Q;W)\leq MC_{f_2}(P\|Q;W) \tag{3.7}$$

where $f_1,\ f_2 : I \in (0,\infty) \subset R_+ \to R,\ \ and\ \ C_{f_1}(P\|Q;W) = \sum_{i=1}^{n} w_i q_i f_i\left(\frac{p_i}{q_i}\right)$.(3.8)

SECTION 4

BOUNDS FOR LOGARITHMIC NON-SYMMETRIC WEIGHTED DIVERGENCE MEASURES

(a) We have the following bounds for D(P || Q ; W)

$$\frac{r^2(r+3)}{(r+1)^3} LS_j(P\|Q;W) \le LN_D(P\|Q;W) \le \frac{R^3(R+3)}{(R+1)^3} LS_J(P\|Q;W)$$

Proof: Setting $p_i = x,\ q_i = 1$ in (2.2), we have the functional form as:

$$f_{LN_D}(x) = (x-1)\log\frac{x+1}{2}, \quad \forall x \in (0,\infty) \tag{4.1}$$

$$\Rightarrow \quad f'_{LN_D}(x) = \frac{x-1}{x+1} + \log\left(\frac{x+1}{2}\right) \tag{4.2}$$

$$\Rightarrow \quad f''_{LN_D}(x) = \frac{(x+3)}{(x+1)^2} \tag{4.3}$$

Also from (2.5), setting $p_i = x,\ q_i = 1$, we have

$$f_{LS_J}(x) = (x-1)\log x, \qquad \forall\ x \in (0,\infty) \tag{4.4}$$

$$\Rightarrow \quad f'_{LS_J}(x) = 1 - \frac{1}{x} + \log x \tag{4.5}$$

$$\Rightarrow \quad f''_{LS_J}(x) = \frac{x+1}{x^2} \tag{4.6}$$

Now let us define

$$g_{DJ}(x) = \frac{f''_{LN_D}(x)}{f''_{LS_J}(x)} = \frac{x^2(x+3)}{(x+1)^3}, \qquad \forall\ x \in (0,\infty) \tag{4.7}$$

$$\Rightarrow \quad g'_{DJ}(x) = \frac{6x}{(x+1)^4} \ge 0, \qquad \forall\ x \in (0,\infty) \tag{4.8}$$

From (4.7), we have

$$m = \inf_{x\in(r,R)} g_{DJ}(x) = \frac{r^2(r+3)}{(r+1)^3} \tag{4.9}$$

and

$$M = \sup_{x\in(r,R)} g_{(DJ)}(x) = \frac{R^2(R+3)}{(R+1)^3} \tag{4.10}$$

Now using (3.7), together with (4.9) and (4.10), we have

$$\frac{r^2(r+3)LS_J}{(r+1)^3}(P\|Q;W) \le LN_D(P\|Q;W) \le \frac{R^2(R+3)}{(R+1)^3} LS_J(P\|Q;W)$$

(b) $$\frac{2r(r+3)LS_I}{r+1}(P\|Q;W) \le LN_D(P\|Q;W) \le \frac{2R(R+3)}{R+1} LS_I(P\|Q;W) \tag{4.11}$$

Proof. Setting $p_i = x,\ q_i = 1$ in (2.6), we have the functional form as:

$$f_{LS_I}(x)=\frac{x}{2}\log\frac{2x}{x+1}+\frac{1}{2}\log\left(\frac{2}{x+1}\right)\forall\ x\in(0,\infty) \tag{4.12}$$

$$\Rightarrow \qquad f'_{LS_I}(x)=\frac{1}{2}\log\frac{2x}{x+1} \tag{4.13}$$

$$\Rightarrow \qquad f''_{LS_I}(x)=\frac{1}{2(x+1)} \tag{4.14}$$

Now we define

$$g_{DI}(x)=\frac{f''_{LN_D}(x)}{f'_{LS_J}(x)}, \qquad \forall\, x\in(0,\infty)$$

$$=\frac{2x(x+3)}{x+1}$$

$$\Rightarrow \qquad g'_{DI}(x)=\frac{2(x^2+2x+3)}{(x+1)^3}>0, \qquad \forall\, x\in(0,\infty). \tag{4.15}$$

Also

$$m=\inf_{x\in(r,R)} g_{DI}(x)=\frac{2r(r+3)}{(r+1)} \tag{4.16}$$

and

$$M=\sup_{x\in(r,R)} g_{DI}(x)=\frac{2R(R+3)}{(R+1)}. \tag{4.17}$$

Now using (3.7) together with (4.16) and (4.17), we get the required bounds i.e.

$$\frac{2r(r+3)}{(r+1)}I(P\,\|\,Q;W)\le LN_D(P\,\|\,Q;W)\le\frac{2R(R+3)}{(R+1)}I(P\,\|\,Q;W).$$

(c) We have the following bound for $LN_F(P \,\|\, Q;\, W)$:

$$0\le LN_F(P\,\|\,Q;W)\le\frac{4}{27}LS_J(P\,\|\,Q;W) \tag{4.18}$$

Proof. First, we have

$$f_{LN_F}(X)=x\log\left(\frac{2x}{x+1}\right), \qquad x\in(0,\infty) \tag{4.19}$$

$$\Rightarrow \qquad f'_{LN_F}(X)=\log\left(\frac{2x}{x+1}\right)+\frac{1}{x+1} \tag{4.20}$$

$$\Rightarrow \qquad f''_{LN_F}(X)=\frac{1}{x(x+1)^2}. \tag{4.21}$$

And $$f_{LS_J}(x)=(x-1)\log x, \qquad \forall\ x\in(0,\infty) \tag{4.22}$$

$$\Rightarrow \qquad f'_{LS_J}(x)=1-\frac{1}{x}+\log x \tag{4.23}$$

and

$$\Rightarrow \qquad f''_{LS_J}(x)=\frac{x+1}{x^2}. \tag{4.24}$$

Now we define

$$g_{FJ}(X)=\frac{f''_{LN_F}(X)}{f''_{LS_J}(X)},\ \forall x\in(0,\infty)=\frac{x}{(x+1)^3} \quad (4.25)$$

$$\Rightarrow \quad g'_{FJ}(X)=-\frac{(2x-1)}{(x+1)^4} \quad (4.26)$$

$$\begin{cases}\geq 0, & x\leq \frac{1}{2}\\ \leq 0, & x\geq \frac{1}{2}\end{cases} \quad (4.26a)$$

From (4.26a), we observe that the function $g_{FJ}(x)$ is increasing in x ∈ (0, ½) and decreasing in x ∈ (1/2; ∞). Hence concave and non-symmetric

Also,

$$M=\sup_{x\in(r,R)} g_{FJ}(x)$$

$$= g_{FJ}\left(\frac{1}{2}\right)$$

$$=\frac{4}{27} \quad (4.27)$$

Now using (3.7) together with (4.27), we get the required bound.

(d) For $LN_F(P\|Q;W)$, we have the following bounds

$$\frac{2}{R+1}LS_I(P\|Q;W)\leq LN_F(P\|Q;W)\leq \frac{2}{r+1}LS_I(P\|Q;W) \quad (4.28)$$

Proof. We have

$$f_{LN_F}(X)=x\log\frac{2x}{(x+1)} \quad \text{(from (4.20))}$$

$$\Rightarrow \quad f''_{LN_F}(X)=\frac{1}{x(x+1)^2} \quad \text{i.e. (4.22)}$$

And

$$f_{LS_I}(x)=\frac{x}{2}\log\left(\frac{2x}{x+1}\right)+\log\left(\frac{2}{x+1}\right),\ \forall\ x\in(0,\infty) \quad (4.29)$$

$$\Rightarrow \quad f'_{LS_I}(x)=\frac{1}{2}\log\left(\frac{2x}{x+1}\right) \quad (4.30)$$

and

$$\Rightarrow \quad f''_{LS_I}(x)=\frac{1}{2x(x+1)} \quad (4.31)$$

Now we define

$$g_{FI}(x)=\frac{g''_{LN_F}(x)}{f''_{LS_I}(x)},\quad \forall x\in(0,\infty)$$

$$= \frac{2}{x+1} \tag{4.32}$$

$$\Rightarrow \qquad g'_{FI}(x) = -\frac{2}{(x+1)^2} < 0, \qquad \forall\, x \in (0,\infty) \tag{4.33}$$

Also

$$\Rightarrow \qquad m = \inf_{x\in(r,R)} g_{FI}(x) = \frac{2}{R+1} \tag{4.34}$$

and

$$M = \sup_{x\in(r,R)} g_{FI}(x) = \frac{2}{r+1} \tag{4.35}$$

Now using (3.7) together with (4.34) and (4.35), we get the required bounds for $LN_F(P\|Q;W)$.

(e) $$\frac{1}{2(R+1)^2} LS_J(P\|Q;W) \le LN_G(P\|Q;W) \le \frac{1}{2(r+1)^2} LS_J(P\|Q) \tag{4.36}$$

Proof. We have

$$f_{LN_G}(X) = \frac{(x+1)}{2}\log\left(\frac{x+1}{2x}\right), \qquad \forall\, x \in (0,\ \infty) \tag{4.37}$$

$$\Rightarrow \qquad f'_{LN_G}(X) = \frac{1}{2}\left[\log\left(\frac{x+1}{2x}\right) - \frac{1}{x}\right] \tag{4.38}$$

and

$$\Rightarrow \qquad f''_{LN_G}(X) = \frac{1}{2x^2(x+1)} \tag{4.39}$$

We have

$$f_{LS_J}(x) = (x-1)\log x \tag{4.40}$$

$$\Rightarrow \qquad f'_{LS_J}(x) = 1 - \frac{1}{x} + \log x \tag{4.41}$$

$$\Rightarrow \qquad f''_{LS_J}(x) = \frac{x+1}{x^2} \tag{4.42}$$

Now we define

$$g_{GJ}(x) = \frac{f''_{LN_G}(x)}{f''_{LS_J}(x)}, \qquad \forall\ x \in (0,\infty)$$

$$= \frac{1}{2(x+1)^2} \tag{4.43}$$

$$\Rightarrow \qquad g'_{GJ}(x) = -\frac{1}{(x+1)^3} < 0, \qquad \forall\ x \in (0,\infty) \tag{4.44}$$

Also

$$m = \inf_{x\in(r,R)} g_{GJ}(x) = \frac{1}{2(R+1)^2} \tag{4.45}$$

and

$$M = \sup_{x\in(r,R)} g_{GJ}(x) = \frac{1}{2(r+1)^2} \tag{4.46}$$

Now using (3.7) together with (4.45) and (4.46) we get the required bound.

(f) We have the following bound $LN_G(P||Q; W)$ in terms of $LS_I(P \,||\, Q; W)$

$$\frac{1}{R} LS_I(P\,||\,Q;W) \le LN_G(P\,||\,Q;W) \le \frac{1}{r} LS_I(P\,||\,Q;W) \tag{4.47}$$

Proof. We havefrom (4.40) and(4.32)

$$f''_{LN_G}(x) = \frac{1}{2x^2(x+1)}$$

and

$$f''_{LN_I}(x) = \frac{1}{2x(x+1)}$$

respectively.

We define

$$g_{GI}(x) = \frac{f''_{LN_G}(x)}{f''_{LN_I}(x)}$$

$$= \frac{1}{x} \tag{4.48}$$

$$\Rightarrow \qquad g'_{GI}(x) = -\frac{1}{x^2} \tag{4.49}$$

Also

$$m = \inf_{x\in(r,R)} g_{GI}(x) = \frac{1}{R} \tag{4.50}$$

and

$$M = \sup_{x\in(r,R)} g_{GI}(x) = \frac{1}{r} \tag{4.51}$$

Using (3.7) together with (4.50) and (4.51), we have the required bound.

(g) We have the following bound for $LN_G(P||Q; W)$ in terms of $LS_T(P\,||\,Q;W)$

$$\frac{2}{1+R^2} LS_T(P\,||\,Q;W) \le LN_G(P\,||\,Q;W) \le \frac{2}{r+1} LS_T(P\,||\,Q;W) \tag{4.52}$$

Proof. We have from (4.39)

$$f''_{LN_G}(x) = \frac{1}{2x^2(x+1)} \tag{4.53}$$

and from (2.7), setting $p_i = x,\ q_i = 1,$ in the functional form

$$f_{LS_T}(x) = \left(\frac{x+1}{2}\right)\log\left(\frac{x+1}{2\sqrt{x}}\right)$$

$$\Rightarrow \qquad f'_{LS_T}(x) = \frac{1}{4}\left[1 - \frac{1}{x} + 2\log\left(\frac{x+1}{2\sqrt{x}}\right)\right] \tag{4.54}$$

$$\Rightarrow \qquad f''_{LS_T}(x) = \frac{1}{4}\left(\frac{1+x^2}{x^2+x^3}\right) = \frac{1}{4}\frac{1}{x^2}\frac{(1+x^2)}{(1+x)} \tag{4.55}$$

Let us define

$$g_{GT}(x) = \frac{f''_{LN_G}(x)}{f'_{LS_T}(x)} = \frac{2}{1+x^2}, \ \ \forall \ x \in (0, \ \infty) \tag{4.56}$$

$$\Rightarrow \qquad g'_{GT}(x) = -\frac{4x}{(1+x^2)^2} < 0, \qquad \forall \ x \in (0, \ \infty) \tag{4.57}$$

Also

$$m = \inf_{x\in(r,R)} g_{GT}(x) = \frac{2}{1+R^2} \tag{4.58}$$

and

$$M = \sup_{x\in(r,R)} g_{GT}(x) = \frac{2}{1+r^2} \tag{4.59}$$

Now using (3.7) together with (4.58) and (4.59) we get the required bounds for $LN_G(P\|Q; W)$.

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