Generalization of fuzzy b-open sets in fuzzy topological spaces

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Abstract

In this paper we introduce the concept of fuzzy ℭ–b-open and fuzzy ℭ–b-closed sets by using arbitrary complement function ℭ and by using fuzzy ℭ - closure operators of a fuzzy topological space where ℭ :[0, 1]→ [0, 1] is a function and investigate some of their basic properties of a fuzzy topological space.

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1. Introduction.

The concept of complement function is used to define a fuzzy closed subset of a fuzzy topological space. That is a fuzzy subset λ is fuzzy closed if the standard complement $1-\lambda = \lambda'$ is fuzzy open. Here the standard complement is obtained by using the function ℭ : [0, 1]→ [0, 1] defined by ℭ (x) = 1–x, for all $x \in [0, 1]$. Several fuzzy topologists used this type of complement while extending the concepts in general topological spaces to fuzzy topological spaces. But there are other complements in the fuzzy literature [8]. This motivated the author to introduce the concepts of fuzzy ℭ-closed sets, fuzzy ℭ-semi closed set, fuzzy ℭ–pre closed set, fuzzy ℭ–α- closed set and fuzzy ℭ–semi pre closed set in fuzzy topological spaces, where ℭ:[0, 1]→ [0, 1] is an arbitrary complement function.

The purpose of this paper is to introduce the concept of fuzzy ℭ-b-open sets where ℭ: [0,1] → [0,1] is an arbitrary complement function. The concept of b-open set in general topology was introduced and studied by Andrijevic in [2]. S.S. Benchalli anf Jenifer J. karnel[] extend this concept of b-open set to fuzzy topological spaces. I have also extend this as, replace the standard complement function with arbitrary complement function in fuzzy topological spaces in this paper. Arbitrary complement function is any function that satisfying the boundary, monotonic, involutive and continuous functions. Georage J. Klir and Bo Yuan said that if the function that satisfying the monotonic and involutive conditions then automatically it satisfies the remaining conditions. In their [ ] investigation, they use the standard complement function ℭ (x) = 1–x, $0 \le x \le 1$.

2. PRELIMINARIES

Throughout this paper (X,τ) denotes a fuzzy topological space in the sense of Chang. Let $\mathfrak{C}: [0, 1]\to[0, 1]$ be a complement function. If λ is a fuzzy subset of (X,τ) then the complement $\mathfrak{C}\lambda$ of a fuzzy subset λ is defined by $\mathfrak{C}\lambda(x) = \mathfrak{C}(\lambda(x))$ for all $x\in X$. A complement function $\mathfrak{C}$ is said to satisfy

(i) the boundary condition if $\mathfrak{C}(0) = 1$ and $\mathfrak{C}(1) = 0$,
(ii) monotonic condition if $x \leq y \Rightarrow \mathfrak{C}(x) \geq \mathfrak{C}(y)$, for all $x, y \in [0, 1]$,
(iii) involutive condition if $\mathfrak{C}(\mathfrak{C}(x)) = x$, for all $x \in[0, 1]$.

The properties of fuzzy complement function $\mathfrak{C}$ and $\mathfrak{C}\lambda$ are given in George Klir[8] and Bageerathi et al[2]. The following lemma will be useful in sequel.

Lemma 2.1 [2]

Let $\mathfrak{C} : [0, 1] \to [0, 1]$ be a complement function that satisfies the monotonic and involutive conditions. Then for any family $\{\lambda_\alpha: \alpha\in\Delta \}$ of fuzzy subsets of X, we have

(i) $\mathfrak{C}(\sup\{\lambda_\alpha(x): \alpha\in\Delta\}) = \inf\{ \mathfrak{C}(\lambda_\alpha(x)): \alpha\in\Delta\}= \inf\{(\mathfrak{C}\lambda_\alpha(x)): \alpha\in\Delta\}$ and

(ii) $\mathfrak{C}(\inf\{\lambda_\alpha(x): \alpha\in\Delta\}) = \sup\{ \mathfrak{C}(\lambda_\alpha(x)): \alpha\in\Delta\}= \sup\{(\mathfrak{C}\lambda_\alpha(x)): \alpha\in\Delta\}$ for $x\in X$.

Definition 2.2 [2]

A fuzzy subset λ of X is fuzzy $\mathfrak{C}$-closed in (X,τ) if $\mathfrak{C}\lambda$ is fuzzy open in (X,τ). The fuzzy $\mathfrak{C}$-closure of λ is defined as the intersection of all fuzzy $\mathfrak{C}$-closed sets μ containing λ. The fuzzy $\mathfrak{C}$-closure of λ is denoted by $cl_{\mathfrak{C}}\lambda$ that is equal to $\wedge\{\mu: \mu \geq \lambda, \mathfrak{C}\mu\in\tau\}$.

Lemma 2.3 [2]

If the complement function C satisfies the monotonic and involutive conditions, then for any fuzzy subset λ of X,

(i) $\mathfrak{C}(int\,\lambda) = cl_{\mathfrak{C}}(\mathfrak{C}\lambda)$ and $\mathfrak{C}(cl_{\mathfrak{C}}\lambda) = int(\mathfrak{C}\lambda)$.
(ii) $\lambda \leq cl_{\mathfrak{C}}\lambda$,
(iii) λ is fuzzy $\mathfrak{C}$-closed $\Leftrightarrow cl_{\mathfrak{C}}\lambda= \lambda$,
(iv) $cl_{\mathfrak{C}}(cl_{\mathfrak{C}}\lambda) = cl_{\mathfrak{C}}\lambda$,
(v) If $\lambda \leq \mu$ then $cl_{\mathfrak{C}}\lambda \leq cl_{\mathfrak{C}}\mu$,
(vi) $cl_{\mathfrak{C}}(\lambda\vee \mu) = cl_{\mathfrak{C}}\lambda \vee cl_{\mathfrak{C}}\mu$,
(vii) $cl_{\mathfrak{C}}(\lambda\wedge \mu) \leq cl_{\mathfrak{C}}\lambda \wedge cl_{\mathfrak{C}}\mu$.
(viii) For any family $\{\lambda_\alpha\}$ of fuzzy sub sets of a fuzzy topological space we have
$\vee\, cl_{\mathfrak{C}}\lambda_\alpha \leq cl_{\mathfrak{C}}(\vee\lambda_\alpha)$ and $cl_{\mathfrak{C}}(\wedge\lambda_\alpha) \leq \wedge\, cl_{\mathfrak{C}}\lambda_\alpha$.

Lemma 2.4 [2]

Let (X,τ) be a fuzzy topological space. Let $\mathfrak{C}$ be a complement function that satisfies the boundary, monotonic and involutive conditions. Then the following conditions hold.

(i) 0 and 1 are fuzzy $\mathfrak{C}$ -closed sets,
(ii) arbitrary intersection of fuzzy $\mathfrak{C}$ -closed sets is fuzzy $\mathfrak{C}$ -closed and
(iii) finite union of fuzzy $\mathfrak{C}$ -closed sets is fuzzy $\mathfrak{C}$ -closed.
(iv) for any family $\{\lambda_\alpha : \alpha \in \Delta\}$ of fuzzy subsets of X. we have
$\mathfrak{C}(\vee\{\lambda_\alpha: \alpha\in\Delta\}) = \wedge\{\mathfrak{C}\lambda_\alpha: \alpha\in\Delta\}$ and $\mathfrak{C}(\wedge\{\lambda_\alpha: \alpha\in\Delta\}) = \vee\{\mathfrak{C}\lambda_\alpha: \alpha\in\Delta\}$.

Definition 2.5 [Definition 2.15, [3]]

A fuzzy topological space (X,τ) is $\mathfrak{C}$ -product related to another fuzzy topological space (Y,σ) if for any fuzzy subset ν of X and ζ of Y, whenever $\mathfrak{C}\lambda \not\geq \nu$ and $\mathfrak{C}\mu \not\geq \zeta$ imply $\mathfrak{C}\lambda \times 1 \vee 1 \times \mathfrak{C}\mu \geq \nu \times \zeta$, where $\lambda \in \tau$ and $\mu \in \sigma$, there exist $\lambda_1 \in \tau$ and $\mu_1 \in \sigma$ such that $\mathfrak{C}\lambda_1 \geq \nu$ or $\mathfrak{C}\mu_1 \geq \zeta$ and $\mathfrak{C}\lambda_1 \times 1 \vee 1 \times \mathfrak{C}\mu_1 = \mathfrak{C}\lambda \times 1 \vee 1 \times \mathfrak{C}\mu$.

Lemma 2.6 [Theorem 2.19, [3]]

Let (X,τ) and (Y,σ) be $\mathfrak{C}$ -product related fuzzy topological spaces. Then for a fuzzy subset λ of X and a fuzzy subset μ of Y, $cl_{\mathfrak{C}}(\lambda \times \mu) = cl_{\mathfrak{C}}\lambda \times cl_{\mathfrak{C}}\mu$.

3. Characterizations of fuzzy b-open sets

In this section the class of fuzzy $\mathfrak{C}$-b- open sets is introduce in a fuzzy topological space using fuzzy $\mathfrak{C}$ -Closure operator.

Definition 3.1

Let (X,τ) be a fuzzy topological space and $\mathfrak{C}$ be a complement function. Then a fuzzy subset λ of X is fuzzy $\mathfrak{C}$ -b- open in (X,τ) if $\lambda \leq Cl_{\mathfrak{C}}\, Int(\lambda) \vee Int\, Cl_{\mathfrak{C}}(\lambda)$.

The class of all fuzzy $\mathfrak{C}$ -b- open sets coincides with the class of all fuzzy b- open sets if $\mathfrak{C}(x) = 1-x$.

The class of all fuzzy b-open sets is denoted by FBO(X). It easy to see that FPO(X) $\vee$ FSO(X) $\subseteq$ FBO{X) $\subseteq$ FSPO{X).For if $\lambda \in$ FPO (X), then by definition $\lambda \leq Int\, Cl_{\mathfrak{C}}(\lambda)$ implies $\lambda \leq Int\, Cl_{\mathfrak{C}}(\lambda) \vee Cl_{\mathfrak{C}}\, Int(\lambda)$. This implies that $\lambda \in$ FBO (X). Now let $\lambda \in$ FSO (X). Then by definition $\lambda \leq Cl_{\mathfrak{C}}\, Int\, \lambda \vee Int\, Cl_{\mathfrak{C}}\, \lambda$ implies that $\lambda \in$ FBO {X) . Finally let $\lambda \in$ FBO (A). Then by definition $\lambda \leq Cl_{\mathfrak{C}} Int\, \lambda \vee Int\, Cl_{\mathfrak{C}}\, \lambda$. And $\lambda \leq Cl_{\mathfrak{C}}\, Int\, \lambda \vee Int\, Cl_{\mathfrak{C}}\, \lambda \leq Cl_{\mathfrak{C}}\, Int\, Cl_{\mathfrak{C}}\, \lambda \vee Int\, Cl_{\mathfrak{C}}\, \lambda \leq Cl_{\mathfrak{C}}\, Int\, Cl_{\mathfrak{C}}\, \lambda$.This implies $\lambda \in$ FSPO (X).

Remark 3. 2

It is clear that every fuzzy ℭ–semi open set is fuzzy ℭ –b-open set and fuzzy ℭ –pre open set is fuzzy ℭ–b-open. But the separate converses are not true as shown by the following example.

Example 3.3

Let X = {a, b} and τ = **{**0, {$a_{.3}$, $b_{.8}$}, {$a_{.2}$, $b_{.5}$}, {$a_{.7}$, $b_{.05}$}, {$a_{.3}$, $b_{.5}$}, {$a_{.3}$, $b_{.05}$}, {$a_{.2}$, $b_{.05}$}, {$a_{.7}$, $b_{.8}$}, {$a_{.7}$, $b_{.5}$}, 1**}**. Let ℭ (x) = $\frac{1-x}{1+2x}$, $0 \le x \le 1$, be the complement function. We see that this complement function satisfying the monotonic and involutive conditions. The family of all fuzzy ℭ -closed sets ℭ (τ) = **{**0, {$a_{.4375}$, $b_{.077}$}, {$a_{.57}$, $b_{.25}$}, {$a_{.125}$, $b_{.86}$}, {$a_{.4375}$, $b_{.25}$}, {$a_{.4375}$, $b_{.86}$}, {$a_{.57}$, $b_{.86}$}, {$a_{.125}$, $b_{.077}$}, {$a_{.125}$, $b_{.25}$}, 1**}**. Let λ = {$a_{.3}$, $b_{.4}$}.Then it can be computed that

$Cl_{\mathfrak{C}}\lambda$ = {$a_{.4375}$, $b_{.86}$} and $Int\, Cl_{\mathfrak{C}}\lambda$ = {$a_{.3}$, $b_{.8}$}

and $Int\, \lambda$ ={$a_{.3}$, $b_{.05}$} and $Cl_{\mathfrak{C}}\, Int\, \lambda$ ={$a_{.4375}$, $b_{.077}$} .

Thus λ = {$a_{.3}$, $b_{.4}$} $\le Cl_{\mathfrak{C}}\, Int\, \lambda \vee Int\, Cl_{\mathfrak{C}}\lambda$ = {$a_{.4375}$, $b_{.86}$}.

By using Proposition 3.2, we see that λ is fuzzy ℭ –b-open.

Also we see that implies $\lambda \not\leq Cl_{\mathfrak{C}}\, Int\, \lambda$ ={$a_{.4375}$, $b_{.077}$}.

That shows, by using Lemma 2.19, we see that λ is not fuzzy ℭ -semi open.

Example 3.4

From Example 3.3, let X = {a, b} and τ = **{**0, {$a_{.3}$, $b_{.8}$}, {$a_{.2}$, $b_{.5}$}, {$a_{.7}$, $b_{.05}$}, {$a_{.3}$, $b_{.5}$}, {$a_{.3}$, $b_{.05}$}, {$a_{.2}$, $b_{.05}$}, {$a_{.7}$, $b_{.8}$}, {$a_{.7}$, $b_{.5}$}, 1**}**. Let λ = {$a_{.4}$, $b_{.8}$}.Then it can be computed that $Cl_{\mathfrak{C}}\lambda$ = {$a_{.57}$, $b_{.86}$} and $Int\, Cl_{\mathfrak{C}}\lambda$ = {$a_{.3}$, $b_{.8}$}
and $Int\, \lambda$ ={$a_{.3}$, $b_{.8}$} and $Cl_{\mathfrak{C}}\, Int\, \lambda$ ={$a_{.4375}$, $b_{.86}$} .

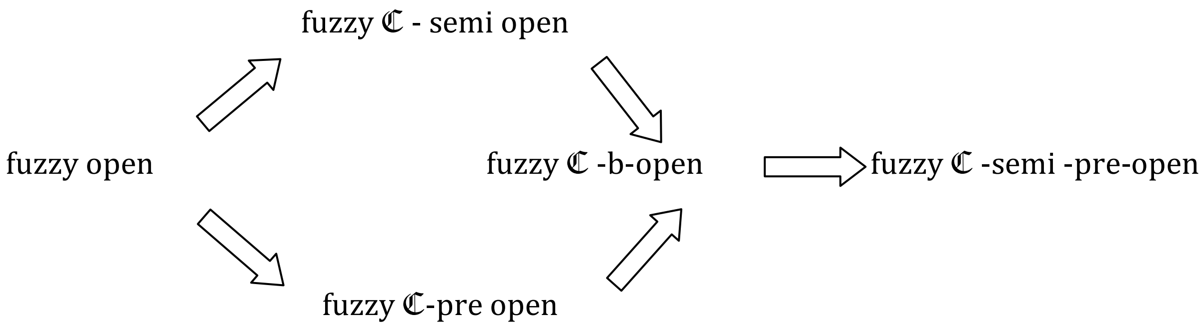
Thus λ = {$a_{.4}$, $b_{.8}$} $\le Cl_{\mathfrak{C}}\, Int\, \lambda \vee Int\, Cl_{\mathfrak{C}}\lambda$ = {$a_{.4375}$, $b_{.86}$}.

By using Proposition 3.2, we see that λ is fuzzy ℭ –b-open.

Also we see that λ = {$a_{.4}$, $b_{.8}$} $\not\leq Int\, Cl_{\mathfrak{C}}\lambda$ = {$a_{.3}$, $b_{.8}$}.

That shows, by Lemma 2.14, we see that λ is not fuzzy ℭ -pre open.

The following diagram of implications is true.



S.S. Benchalli and Jenifer J. Karnel **[9]** established that the intersection of fuzzy b-open sets is not a fuzzy b-open set. The next example shows that the intersection of any two fuzzy ℭ-b-open sets is not fuzzy ℭ - b-open.

Example 3.5

From Example 3.3, let $X = \{a, b\}$ and $\tau = \{0, \{a_{.3}, b_{.8}\}, \{a_{.2}, b_{.5}\}, \{a_{.7}, b_{.05}\}, \{a_{.3}, b_{.5}\}, \{a_{.3}, b_{.05}\}, \{a_{.2}, b_{.05}\}, \{a_{.7}, b_{.8}\}, \{a_{.7}, b_{.5}\}, 1\}$. Let $\lambda = \{a_{.2}, b_{.6}\}$, it can be found that $Cl_{C}\,\lambda = \{a_{.4375}, b_{.86}\}$, $Int\,Cl_{C}\,\lambda = \{a_{.3}, b_{.8}\}$ and $Int\ \lambda = \{a_{.2}, b_{.5}\}$ and $Cl_{\mathfrak{C}}\,Int\,\lambda = \{a_{.4375}, b_{.86}\}$.

Thus $\lambda = \{a_{.2}, b_{.6}\} \le Cl_{\mathfrak{C}}\,Int\,\lambda \vee Int\,Cl_{\mathfrak{C}}\,\lambda = \{a_{.4375}, b_{.86}\}$.

By using Proposition 3.2, we see that λ is fuzzy ℭ –b-open.

And let $\mu = \{a_{.8}, b_{.25}\}$, it follows that $Cl_{\mathfrak{C}}\,\mu = \{1\}$, $Int\,Cl_{\mathfrak{C}}\,\mu = \{1\}$

and $Int\ \mu = \{a_{.7}, b_{.05}\}$ and $Cl_{\mathfrak{C}}\,Int\,\mu = \{1\}$.

Thus $\mu = \{a_{.4375}, b_{.86}\} \le Cl_{\mathfrak{C}}\,Int\,\mu \vee Int\,Cl_{\mathfrak{C}}\,\mu = \{1\}$.

By using Proposition 3.2, we see that μ is fuzzy ℭ –b-open.

Now $\lambda\wedge\mu = \{a_{.2}, b_{.25}\}$, $Cl_{\mathfrak{C}}\,(\lambda\wedge\mu) = \{a_{.4375}, b_{.25}\}$, $Int\,Cl_{\mathfrak{C}}\,(\lambda\wedge\mu) = \{a_{.3}, b_{.05}\}$

and $Int\ \lambda\wedge\mu = \{a_{.2}, b_{.05}\}$ and $Cl_{\mathfrak{C}}\,Int\,\lambda\wedge\mu = \{ a_{.4375}, b_{.076}\}$.

Thus $\lambda\wedge\mu = \{a_{.4375}, b_{.86}\} \not\le Cl_{\mathfrak{C}}\,Int\,(\lambda\wedge\mu) \vee Int\,Cl_{\mathfrak{C}}\,(\lambda\wedge\mu) = \{\{ a_{.4375}, b_{.076}\}\}$.

By using Proposition 3.2, we see that $\lambda\wedge\mu$ is not fuzzy ℭ –b-open even though λ and μ are fuzzy ℭ -b- open.

S.S. Benchalli and Jenifer J. Karnel **[9]** established that any union of fuzzy b- open sets is a fuzzy b- open set. The next example shows that the union of any two fuzzy ℭ-b-open sets is not fuzzy ℭ -b- open.

Example 2.6

Let X = {a, b, c} and τ = **{**0, {a.6, b.3}, {b.4, c.6}, {a.2, c.5}, {b.3}, {a.6, b.4, c.6}, {a.2, b.4, c.6}, {c.5}, {a.2}, {a.6, b.3, c.5}, {a.2, b.3}, {a.2, b.3, c.5}, {b.3, c.5}, 1**}**. Then (X, τ) is a fuzzy topological space. Let ℭ (x) = $\frac{2x}{1+x}$, 0 ≤ x ≤ 1, be a complement function and ℭ does not satisfy the monotonic and involutive conditions. The family of all fuzzy ℭ-closed sets ℭ(τ) = **{**0, {a.75, b.46}, {b.571, c.75}, {a.33, c.667}, {b.462}, {a.75, b.571, c.75}, {a.33, b.57, c.75}, {c.667}, {a.33}, {a.75, b.462, c.667}, {a.33, b.462}, {a.33, b.462, c.667}, {b.462, c.667}, 1**}**.

Let λ = {a.75, b.35}, it can be find that $Cl_{\mathfrak{C}}$ λ = {a.75, b.46}, $Int\ Cl_{\mathfrak{C}}$ λ = {a.6, b.3}.

Int λ= {a.6, b.3} and $Cl_{\mathfrak{C}}\ Int$ λ= {a.75, b.46}.

Thus λ = {a.75, b.35} ≤ $Cl_{\mathfrak{C}}\ Int\ \lambda \vee Int\ Cl_{\mathfrak{C}}\ \lambda$ = {a. 75, b.46}.

By using Proposition 3.2, we see that λ is fuzzy ℭ –b-open.

Let μ = {b.45, c.75}, it follows that $Cl_{\mathfrak{C}}$ μ = {b.57, c.75}, $Int\ Cl_{\mathfrak{C}}$ μ = {b.4, c.6}

Int μ= {b.4, c.6} and $Cl_{\mathfrak{C}}\ Int$ μ= {b.571, b.75}.

Thus μ = { b.45, c.75} ≤ $Cl_{\mathfrak{C}}\ Int\ \mu \vee Int\ Cl_{\mathfrak{C}}\ \mu$ = {a. 571, b.75}.

By using Proposition 3.2, we see that λ is fuzzy ℭ –b-open.

Now λ∨μ = {a.75, b.45, c.75}, $Cl_{\mathfrak{C}}$ (λ∨ μ) = {a.75, b.57, c.75} and $Int\ Cl_{\mathfrak{C}}$ (λ∨μ) = {a.6, b.4, c.6} Int (λ∨ μ)= {a.6, b.4, c.6} and $Cl_{\mathfrak{C}}\ Int$ (λ∨ μ)= {a.75, b.462, c.667}.

Thus (λ∨ μ) = {a.75, b.45, c.75} ≰ $Cl_{\mathfrak{C}}\ Int\ (\lambda \vee \mu) \vee Int\ Cl_{\mathfrak{C}}\ (\lambda \vee \mu)$ = {a.75, b.462, c.667}.

By using Proposition 3.2, we see that λ is not fuzzy ℭ –b-open.

By using Proposition 3.2, λ∨μ is not fuzzy ℭ-b- open, even though λ and μ are fuzzy ℭ-b-open.

If the complement function ℭ satisfies the monotonic and involutive conditions, then union of two fuzzy ℭ-b-open sets is again fuzzy ℭ-b-open as shown in the next proposition.

Theorem 3.7

Let (X, τ) be a fuzzy topological space and ℭ be a complement function that satisfies the monotonic and involutive conditions. Then the arbitrary union of fuzzy ℭ -b- open sets is fuzzy ℭ-b- open.

Proof

Let $\{\lambda_\alpha\}$ be a collection of fuzzy ℭ-b-open sets of a fuzzy space X. Then for each α, $\vee\lambda_\alpha \leq \vee(\text{Int } Cl_{\mathfrak{C}}(\lambda_\alpha) \vee Cl_{\mathfrak{C}} \text{Int}(\lambda_\alpha))$. Since ℭ satisfies the monotonic and involutive properties, by using Lemma 2.8, we have $\vee Cl_{\mathfrak{C}}(\mu_\alpha) \leq Cl_{\mathfrak{C}}(\vee\mu_\alpha)$. By using Lemma 2.17, we have arbitrary union of fuzzy open sets is fuzzy open, that implies $\vee Int(\lambda_\alpha) \leq Int \vee (\lambda_\alpha)$. Finally, $\vee\lambda_\alpha \leq \text{Int } Cl_{\mathfrak{C}}(\vee\lambda_\alpha) \vee Cl_{\mathfrak{C}} \text{Int}(\vee\lambda_\alpha))$.By using Definition 3.1, we have $\{\vee\lambda_\alpha\}$ is a fuzzy ℭ–b-open set.

Theorem 3.8

Let (X, τ) and (Y, σ) be ℭ -product related fuzzy topological spaces. Then the product $\lambda_1 \times \lambda_2$ of a fuzzy ℭ -b- open set λ_1 of X and a fuzzy ℭ -b- open set λ_2 of Y is a fuzzy ℭ -b-open set of the fuzzy product space X × Y.

Proof.

Let λ_1 be a fuzzy ℭ-b- open subset of X and λ_2 be a fuzzy ℭ-b- open subset of Y. Then by using Definition 3.1, $\lambda_1 \leq \text{Int } Cl_{\mathfrak{C}}(\lambda_1) \vee Cl_{\mathfrak{C}} \text{Int}(\lambda_1)$ and $\lambda_2 \leq \text{Int } Cl_{\mathfrak{C}}(\lambda_2) \vee Cl_{\mathfrak{C}} \text{Int}(\lambda_2))$. Using Lemma ……., That implies $\lambda_1 \times \lambda_2 \leq \text{Int } Cl_{\mathfrak{C}}(\lambda_1) \vee Cl_{\mathfrak{C}} \text{Int}(\lambda_1) \times \text{Int } Cl_{\mathfrak{C}}(\lambda_2) \vee Cl_{\mathfrak{C}} \text{Int}(\lambda_2)$. By applying Lemma 2.12, $\lambda_1 \times \lambda_2 \leq \text{Int } Cl_{\mathfrak{C}}(\lambda_1 \times \lambda_2) \vee Cl_{\mathfrak{C}} \text{Int}(\lambda_1 \times \lambda_2)$. Again by using Definition 3.1, $\lambda_1 \times \lambda_2$ is a fuzzy ℭ –b-open set of the fuzzy product space X × Y.

4. Fuzzy ℭ -b- closed sets

This section is devoted to the concept of fuzzy ℭ -b- closed sets that are defined by using fuzzy interior and fuzzy ℭ -closure operator.

Definition 4.1

Let (X,τ) be a fuzzy topological space and ℭ be a complement function. Then a fuzzy subset λ of X is called fuzzy ℭ -b- closed in (X,τ) if $\lambda > cl_{\mathfrak{C}}\, int(\lambda) \wedge int\, cl_{\mathfrak{C}}(\lambda)$.

Remark 4.2

If ℭ (x) = 1– x, then $\lambda \leq Cl\, Int(\lambda) \vee Int\, Cl(\lambda)$.

$$\Rightarrow 1 - \lambda \geq 1 - (Cl\, Int(\lambda)) \wedge 1\text{-}(Int\, Cl(\lambda))$$

$\Rightarrow \lambda^c \geq int\, Cl\, (\lambda^c)) \wedge Cl\, int\, (\lambda^c))$

So, the class of all fuzzy ℭ -b- closed sets coincides with the class of all fuzzy b-closed sets if ℭ (x) = 1–x.

The standard complement of fuzzy b- open is fuzzy b- closed. The analogous result is not true for fuzzy ℭ -b- open. If the complement function ℭ satisfies the monotonic and involutive condition, then the arbitrary complement of fuzzy ℭ -b- open is fuzzy ℭ -b- closed.

Proposition 4.3

Let (X, τ) be a fuzzy topological space and ℭ be a complement function. Then λ is fuzzy ℭ -b- closed if and only if ℭ λ is fuzzy ℭ -b- open if ℭ satisfies the monotonic and involutive conditions.

Proof.

Let λ be fuzzy ℭ -b- closed. Then by using Definition 4.1, $\lambda \geq cl_{\mathfrak{C}}\, int\, (\lambda) \wedge int\, cl_{\mathfrak{C}}\, (\lambda)$. Taking arbitrary complement on both sides, we get $\mathfrak{C}\, \lambda \leq \mathfrak{C}\, (cl_{\mathfrak{C}}\, int\, (\lambda) \vee int\, cl_{\mathfrak{C}}\, (\lambda))$. Since ℭ satisfies the monotonic and involutive conditions, $\mathfrak{C}\, \lambda \leq int\, cl_{\mathfrak{C}}\, (\mathfrak{C}\, \lambda) \wedge cl_{\mathfrak{C}}\, int\, (\mathfrak{C}\, \lambda)$. By using Definition 3.1, ℭλ is fuzzy ℭ-b- open. Let λ be fuzzy ℭ -b- open. Then by using Definition 3.1, $\mathfrak{C}\, \lambda \leq int\, cl_{\mathfrak{C}}\, (\mathfrak{C}\, \lambda) \wedge cl_{\mathfrak{C}}\, int\, (\mathfrak{C}\, \lambda)$. Taking arbitrary complement on both sides, we get $\mathfrak{C}\, (\mathfrak{C}\, \lambda) \geq \mathfrak{C}\, (int\, cl_{\mathfrak{C}}\, (\mathfrak{C}\lambda)) \vee \mathfrak{C}(cl_{\mathfrak{C}}\, int\, (\mathfrak{C}\, \lambda))$. Since ℭ satisfies the monotonic and involutive condition, $\lambda \leq Cl_{\mathfrak{C}}\, int\, (\lambda) \wedge int\, cl_{\mathfrak{C}}\, (\lambda)$. Thus, λ is fuzzy ℭ -b- closed.

The following example shows that the monotonic and involutive conditions can be dropped then the above proposition is not true.

Example 4.4

Let X = {a, b, c} and τ = **{**0, {$a_{.6}$, $b_{.3}$, $c_{.8}$}, {$a_{.5}$, $b_{.4}$, $c_{.2}$}, {$a_{.5}$, $b_{.3}$, $c_{.2}$}, {$a_{.6}$, $b_{.4}$, $c_{.8}$}, 1**}**. Let C(x) = $\sqrt{x}$, $0 \leq x \leq 1$, be a complement function. We see that the complement function C does not satisfy the monotonic and involutive conditions. Then by Definition 3.1, family of all fuzzy C-closed sets, C(τ) = **{**0^*, {$a_{.775}$, $b_{.548}$, $c_{.894}$}, {$a_{.707}$, $b_{.632}$, $c_{.447}$},{$a_{.707}$, $b_{.548}$, $c_{.447}$}, {$a_{.775}$, $b_{.632}$, $c_{.894}$}, 1^* **}**.

Let λ = {$a_{.7}$, $b_{.5}$, $c_{.3}$}, it can be calculated that *Int* λ = {$a_{.5}$, $b_{.4}$, $c_{.2}$} and Cl_C (*Int* λ) = {$a_{.707}$, $b_{.548}$, $c_{.447}$}. Also $Cl_C\, \lambda$ = {$a_{.707}$, $b_{.548}$, $c_{.447}$}, $Int\, Cl_C\, \lambda$ = {$a_{.5}$, $b_{.4}$, $c_{.2}$}. We see that λ = {$a_{.7}$, $b_{.5}$, $c_{.3}$} $\leq Int\, Cl_C\, \lambda \vee Cl_C\, (Int\, \lambda)$ = {$a_{.707}$, $b_{.548}$, $c_{.447}$}.

Now ℭ λ = {$a_{.837}$, $b_{.707}$, $c_{.548}$}, it can be computed that $Int\, Cl_C\, (\mathfrak{C}\, \lambda)$ = 1 and

Cl_C (Int ($\mathfrak{C}\,\lambda$)) = {a.$_{775}$, b.$_{548}$, c.$_{894}$}. Thus $\mathfrak{C}\,\lambda$ = {a.$_{837}$, b.$_{707}$, c.$_{548}$} $\not\leq$ $Int\, Cl_C\, \mathfrak{C}\,\lambda \vee Cl_C\,(Int\,\mathfrak{C}\,\lambda)$ = {1}. This shows that the monotonic and involutive conditions can be dropped then the above proposition is not true.

S.S. Benchalli and Jenifer J. Karnel **[9]** established that any union of fuzzy b- closed sets is not fuzzy b- closed set. However the following example shows that the union of any two fuzzy $\mathfrak{C}$-b- closed sets is not fuzzy $\mathfrak{C}$-b- closed.

Example 4.5

Let X = {a, b, c} and τ = **{**0, {c.$_4$}, {a.$_7$}, {a.$_7$, c.$_4$}, 1**}**. Let C (x) = $\frac{1-x}{1+2x}$, $0 \leq x \leq 1$, be a complement function. Then the family of all fuzzy C -closed sets is C (τ) = **{**0, {a$_1$, b$_1$, c.$_{33}$}, {a.$_{125}$, b$_1$, c$_1$}, {a.$_{125}$, b$_1$, c.$_{33}$}, 1**}**. Let λ = {c.$_4$}, μ = {a.$_7$} and $\lambda\vee\mu$ = {a.$_7$, c.$_4$}. Then $Int\,\lambda$ = {c.$_4$}, $Cl_{\mathfrak{C}}\, Int\,\lambda$ = {a.$_{125}$, b$_1$, c$_1$} and

$Cl_{\mathfrak{C}}\,\lambda$ = {a.$_{125}$, b$_1$, c$_1$} and $Int\, Cl_{\mathfrak{C}}\,\lambda$ = {c.$_4$}. Thus λ = { c.$_4$} $\geq Int\, Cl_C\,\lambda \wedge Cl_C\,(Int\,\lambda)$ = {a.$_{125}$, b$_0$, c.$_4$}.By using Proposition 4.2, we see that λ is fuzzy $\mathfrak{C}$ –b-open. Now $Int\,\mu$ = {c.$_4$}, $Cl_{\mathfrak{C}}\, Int\,\mu$ = {a.$_{125}$, b$_1$, c$_1$} and $Int\, Cl_{\mathfrak{C}}\, Int\,\mu$ = {c.$_4$} $\leq \mu$. By Proposition 4.4, shows that λ and μ are fuzzy C-b-closed sets. Now, Int ($\lambda\vee\mu$) = {a.$_7$, c.$_4$} $Cl_{\mathfrak{C}}\, Int$ ($\lambda\vee\mu$) = {1} and $Int\, Cl_{\mathfrak{C}}\, Int$ ($\lambda\vee\mu$) = 1$\not\leq$ $\lambda\vee\mu$. By using Proposition 4.4, $\lambda\vee\mu$ is not fuzzy C –b- closed.

S.S. Benchalli and Jenifer J. Karnel **[9]** established that the intersection of fuzzy b- closed sets is fuzzy b- closed. Moreover the following examples shows that the intersection of any two fuzzy $\mathfrak{C}$-b- closed sets is not fuzzy $\mathfrak{C}$-b- closed.

Example 4.6

Let X = {a, b} and τ = {0 , {a.$_3$, b.$_8$}, {a.$_2$, b.$_5$}, {a.$_7$, b.$_1$}, {a.$_3$, b.$_5$}, {a.$_3$, b.$_1$}, {a.$_2$, b.$_1$}, {a.$_7$, b.$_8$}, {a.$_7$, b.$_5$}, 1}. Let $\mathfrak{C}$(x) = $\sqrt{x}$, $0 \leq x \leq 1$ be the complement function. From this example, we see that $\mathfrak{C}$ does not satisfy the monotonic and involutive conditions. The family of all fuzzy $\mathfrak{C}$-closed sets is $\mathfrak{C}(\tau)$ = {0, {a.$_{548}$, b.$_{894}$}, {a.$_{447}$, b.$_{707}$},{a.$_{837}$, b.$_{316}$}, {.$_{548}$, b.$_{707}$},{a.$_{548}$, b.$_{316}$}, {a.$_{447}$, b.$_{316}$},{a.$_{837}$, b.$_{894}$}, {a.$_{837}$, b.$_{707}$},1}. Let λ = {a.$_6$, b.$_3$} and μ = {a.$_2$, b.$_4$}. Then it can be calculated that $Int\,\lambda$ = {a.$_3$, b.$_1$}, $Cl_{\mathfrak{C}}\, Int\,\lambda$ = {a.$_{447}$, b.$_{316}$} and $Cl_{\mathfrak{C}}\,\lambda$ = {a.$_{837}$, b.$_{316}$}, $Int\, Cl_{\mathfrak{C}}\,\lambda$ = {a.$_7$, b.$_1$}. Thus λ = {a.$_6$, b.$_3$} $\geq Cl_{\mathfrak{C}}\, Int\,\lambda \wedge Int\, Cl_{\mathfrak{C}}\,\lambda$ = {a.$_{447}$, b.$_1$}. By using Proposition 4.2, we see that λ is fuzzy $\mathfrak{C}$ –b-closed.

And $Int\,\mu$ = {a.$_2$, b.$_1$}, $Cl_{\mathfrak{C}}\, Int\,\mu$ = {b.$_{446}$, c.$_{707}$} and $Int\, Cl_{\mathfrak{C}}\,\mu$ = {a.$_3$, b.$_5$}. Thus λ = {a.$_2$, b.$_4$} $\geq Cl_{\mathfrak{C}}\, Int\,\lambda \wedge Int\, Cl_{\mathfrak{C}}\,\lambda$ = {a.$_3$, b.$_5$}. By using Proposition 4.2, we see that μ is fuzzy $\mathfrak{C}$ –b-closed. This

shows that λ and μ are fuzzy C-b- closed sets. Now, $Int\,\lambda\wedge\mu$ = { a.$_3$, b.$_5$} and $Cl_{\mathfrak{C}}\, Int\,(\lambda\wedge\mu) \neq$ { a.$_3$, b.$_5$} and $Int\, Cl_{\mathfrak{C}}\,(\lambda\wedge\mu)$ = {a.$_3$, b.$_1$}. Thus ($\lambda\wedge\mu$) = {a.$_3$, b.$_5$} $Cl_{\mathfrak{C}}\, Int\,(\lambda\wedge\mu) \wedge Int\, Cl_{\mathfrak{C}}\,(\lambda\wedge\mu)$ =

{a.3, b.1} $\nleq$ (λ∧μ) = {a.3, b.5}. By using Proposition 4.2, we see that (λ∧μ) is not fuzzy ℭ –b-closed.

Remark 4.7

Further, the Example 4.6 shows that the intersection of any two fuzzy ℭ - b- closed sets is not fuzzy ℭ-b-closed, even though the complement function satisfies the monotonic and involutive conditions.

If the complement function ℭ satisfies the monotonic and involutive conditions. Then arbitrary intersection of fuzzy ℭ-b- closed sets is fuzzy ℭ -b- closed as shown in the following proposition.

Proposition 4.8

Let (X, τ) be a fuzzy topological space and ℭ be a complement function that satisfies the monotonic and involutive conditions. Then arbitrary intersection of fuzzy ℭ -b- closed sets is fuzzy ℭ -b- closed.

Proof.

Let $\{\lambda_\alpha\}$ be a collection of fuzzy ℭ-b-open sets of a fuzzy space X. Then for each α, $\vee\lambda_\alpha \leq \vee(\mathit{Int}\ \mathit{Cl}_{\mathfrak{C}}(\lambda_\alpha) \vee \mathit{Cl}_{\mathfrak{C}}\ \mathit{Int}(\lambda_\alpha))$. Since ℭ satisfies the monotonic and involutive properties, by using Lemma 2.8, we have $\vee \mathit{Cl}_{\mathfrak{C}}(\mu_\alpha) \leq \mathit{Cl}_{\mathfrak{C}}(\vee\mu_\alpha)$. By using Lemma 2.17, we have arbitrary union of fuzzy open sets is fuzzy open, that implies $\vee \mathit{Int}(\lambda_\alpha) = \mathit{Int} \vee (\lambda_\alpha)$. By using Definition 3.1, we have $\{\vee\lambda_\alpha\}$ is a fuzzy ℭ–b- open set.

It is clear that every fuzzy ℭ -semi closed and fuzzy ℭ-pre closed set is fuzzy ℭ -b-closed set. But the converse is not true as shown by the following example.

Example 4.9

Let X = {a,b,c} and τ = **{**0, {a.2, c.5}, {b.3}, {a.2, b.3, c.5}, 1**}**. Let $\mathfrak{C}(x) = \frac{1-x}{1+3x}$, $0 \leq x \leq 1$, be a complement function. Then the family of all fuzzy ℭ -closed sets ℭ (τ) = **{**0, {a.5, b1, c.2}, {a1, b.36, c1}, {a.5, b.36, c.2}, 1**}**. Let λ = {a.2, b.3, c.6}. Then *Int* λ = {a.2, b.3, c.5}, $\mathit{Cl}_{\mathfrak{C}}$ *Int* λ = {a1, b.36, c1} and *Int* $\mathit{Cl}_{\mathfrak{C}}$ λ= {a.2, b.3, c.5}. This implies that $\lambda \geq \mathit{Cl}_{\mathfrak{C}}\ \mathit{Int}\ \lambda$

$\wedge$ *Int Cl*$_{\mathfrak{C}}$ λ = {a.$_2$, b.$_3$, c.$_5$}. By using Proposition 4.2, λ is fuzzy $\mathfrak{C}$-b-closed. Also *Cl*$_{\mathfrak{C}}$ *Int* λ= {a$_1$, b.$_{36}$, c$_1$} $\not\leq$ λ, this shows that λ is not fuzzy $\mathfrak{C}$ -pre closed.

It is clear that every fuzzy $\mathfrak{C}$ -semi closed set is fuzzy $\mathfrak{C}$-b-closed. But the converse is not true as shown in the following example.

Example 4.10

From Example 4.8, let X = {a, b, c} and τ = **{**0, {a.$_2$, c.$_5$}, {b.$_3$}, {a.$_2$, b.$_3$, c.$_5$}, 1**}**. Let λ = {a.$_5$, b.$_2$, c.$_3$}, it can be computed that *Int* λ = {0}, *Cl*$_{\mathfrak{C}}$ *Int* λ = {0} and *Int Cl*$_{\mathfrak{C}}$ λ= {1}. This implies that *Int Cl*$_{\mathfrak{C}}$ $\lambda \wedge$ *Cl*$_{\mathfrak{C}}$ *Int* λ = {0} $\leq \lambda$= {a.$_5$, b.$_2$, c.$_3$}. By using Proposition 4.4, λ is fuzzy $\mathfrak{C}$ -b-closed. Also *Int Cl*$_{\mathfrak{C}}$ λ= {1} $\not\leq$ λ, this shows that λ is not fuzzy $\mathfrak{C}$-semi closed.

Theorem 4.11

Let (X, τ) and (Y, σ) be $\mathfrak{C}$ -product related fuzzy topological spaces. Then the product $\lambda_1 \times \lambda_2$ of a fuzzy $\mathfrak{C}$ -b- closed set λ_1 of X and a fuzzy $\mathfrak{C}$ -b- closed set λ_2 of Y is a fuzzy $\mathfrak{C}$ -b-closed set of the fuzzy product space X $\times$ Y.

Proof.

Let λ_1 be a fuzzy $\mathfrak{C}$-b-closed subset of X and λ_2 be a fuzzy $\mathfrak{C}$ -b- closed subset of Y. Then by using Definition 3.1, $\lambda_1 \geq$ Int *Cl*$_{\mathfrak{C}}$ (λ_1)$\wedge$ *Cl*$_{\mathfrak{C}}$ Int (λ_1) and $\lambda_2 \geq$ *Int Cl*$_{\mathfrak{C}}$ (λ_2) $\wedge$ *Cl*$_{\mathfrak{C}}$ *Int* (λ_2)). That implies $\lambda_1 \times \lambda_2 \geq$ Int *Cl*$_{\mathfrak{C}}$ (λ_1) $\wedge$ *Cl*$_{\mathfrak{C}}$ Int (λ_1) $\times$ *Int Cl*$_{\mathfrak{C}}$ (λ_2)$\vee$ *Cl*$_{\mathfrak{C}}$ *Int* (λ_2). This can be written as $\lambda_1 \times \lambda_2 \geq$ Int *Cl*$_{\mathfrak{C}}$ (λ_1) $\times$ *Int Cl*$_{\mathfrak{C}}$ (λ_2) $\wedge$ *Cl*$_{\mathfrak{C}}$ Int (λ_1) $\times$ *Cl*$_{\mathfrak{C}}$ *Int* (λ_2). By applying Lemma 2.12, $\lambda_1 \times \lambda_2 \geq$ *Int Cl*$_{\mathfrak{C}}$ ($\lambda_1 \times \lambda_2$) $\wedge$ *Cl*$_{\mathfrak{C}}$ *Int* ($\lambda_1 \times \lambda_2$). Again by using Definition 3.1, $\lambda_1 \times \lambda_2$ is a fuzzy $\mathfrak{C}$–b-closed set of the fuzzy product space X $\times$ Y.

5. Fuzzy $\mathfrak{C}$ -b- interior and fuzzy $\mathfrak{C}$ -b- closure

In this section, we define the concept of fuzzy $\mathfrak{C}$ -b- interior and fuzzy $\mathfrak{C}$ -b- closure and investigate some of their basic properties.

Definition 5.1

Let (X,τ) be a fuzzy topological space and $\mathfrak{C}$ be a complement function. Then for a fuzzy subset λ of X, the fuzzy $\mathfrak{C}$ -b- interior of λ (briefly *bInt*$_{\mathfrak{C}}$ λ), is the union of all fuzzy $\mathfrak{C}$-b- open sets of X contained in λ.

That is, *bInt*$_{\mathfrak{C}}$ (λ) = $\vee$ {μ; $\mu \leq \lambda$, μ is fuzzy $\mathfrak{C}$ -b- open}.

Proposition 5.2

Let (X, τ) be a fuzzy topological space and let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involvtive conditions. Then for any fuzzy subsets λ and μ of a fuzzy topological space X, we have

(i) $Int\lambda \leq bInt_{\mathfrak{C}}\lambda$,

(ii) $bInt_{\mathfrak{C}}\lambda$ is fuzzy $\mathfrak{C}$-b- open,

(iii) λ is fuzzy $\mathfrak{C}$ -b- open ⇔ $bInt_{\mathfrak{C}}\lambda = \lambda$,

(iv) $bInt_{\mathfrak{C}}(bInt_{\mathfrak{C}}\lambda) = bInt_{\mathfrak{C}}\lambda$,

(v) If $\lambda \leq \mu$ then $bInt_{\mathfrak{C}}\lambda \leq bInt_{\mathfrak{C}}\mu$.

Proof.

By using Remark, every fuzzy open set is fuzzy $\mathfrak{C}$ -b- open. So, we have $Int\lambda \leq bInt_{\mathfrak{C}}\lambda$. This proves (i).

(ii) follows from Definition 5.1.

Let λ be fuzzy $\mathfrak{C}$ -b- open. Since $\lambda \leq \lambda$, by Definition 5.1, $\lambda \leq bInt_{\mathfrak{C}}\lambda$. By using (ii), we get $bInt_{\mathfrak{C}}\lambda = \lambda$. Conversely we assume that $bInt_{\mathfrak{C}}\lambda = \lambda$. By using Definition 5.1, λ is fuzzy $\mathfrak{C}$ -b-open. Thus (iii) is proved.

By using (iii), we get $bInt_{\mathfrak{C}}(bInt_{\mathfrak{C}}\lambda) = bInt_{\mathfrak{C}}\lambda$. This proves (iv).

Since $\lambda \leq \mu$, by using (i), $bInt_{\mathfrak{C}}\lambda \leq \lambda \leq \mu$. This implies that $bInt_{\mathfrak{C}}(bInt_{\mathfrak{C}}\lambda) \leq bInt_{\mathfrak{C}}\mu$. By using (iii), we get $bInt_{\mathfrak{C}}\lambda \leq bInt_{\mathfrak{C}}\mu$. This proves (v).

Proposition 5.3

Let (X, τ) be a fuzzy topological space and let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then for any two fuzzy subsets λ and μ of a fuzzy topological space, we have (i) $bInt_{\mathfrak{C}}(\lambda \vee \mu) \geq bInt_{\mathfrak{C}}\lambda \vee bInt_{\mathfrak{C}}\mu$ and (ii) $bInt_{\mathfrak{C}}(\lambda \wedge \mu) \leq bInt_{\mathfrak{C}}\lambda \wedge bInt_{\mathfrak{C}}\mu$.

Proof.

Since $\lambda \leq \lambda \vee \mu$ and $\mu \leq \lambda \vee \mu$. By using Proposition 5.2(iv), we get $bInt_{\mathfrak{C}}\lambda \leq bInt_{\mathfrak{C}}(\lambda \vee \mu)$ and $bInt_{\mathfrak{C}}\mu \leq bInt_{\mathfrak{C}}(\lambda \vee \mu)$. This implies that $bInt_{\mathfrak{C}}\lambda \vee bInt_{\mathfrak{C}}\mu \leq bInt_{\mathfrak{C}}(\lambda \vee \mu)$.

Since $\lambda \wedge \mu \leq \lambda$ and $\lambda \wedge \mu \leq \mu$. By using Proposition 5.2(v), we get $bInt_{\mathfrak{C}}(\lambda\wedge\mu) \leq bInt_{\mathfrak{C}}\,\lambda$ and $bInt_{\mathfrak{C}}(\lambda\wedge\mu) \leq bInt_{\mathfrak{C}}\,\mu$. This implies that $bInt_{\mathfrak{C}}(\lambda\wedge\mu) \leq bInt_{\mathfrak{C}}\,\lambda \wedge bInt_{\mathfrak{C}}\,\mu$.

Definition 5.4

Let (X,τ) be a fuzzy topological space. Then for a fuzzy subset λ of X, the fuzzy ℭ-b-closure of λ (briefly $bCl_{\mathfrak{C}}\,\lambda$), is the intersection of all fuzzy ℭ-b- closed sets containing λ.

That is $bCl_{\mathfrak{C}}\,\lambda = \wedge\{\mu: \mu \geq \lambda, \mu$ is fuzzy ℭ-b-closed$\}$.

The concepts of "fuzzy ℭ - b- closure" and "fuzzy b- closure" are identical if ℭ is the standard complement function.

Proposition 5.5

If the complement functions ℭ satisfies the monotonic and involutive conditions. Then for any fuzzy subset λ of X, (i) $\mathfrak{C}\,(bInt_{\mathfrak{C}}\,\lambda) = bCl_{\mathfrak{C}}\,(\mathfrak{C}\,\lambda)$ and (ii) $\mathfrak{C}\,(bCl_{\mathfrak{C}}\,\lambda) = bInt_{\mathfrak{C}}(\mathfrak{C}\lambda)$, where $bInt_{\mathfrak{C}}\,\lambda$ is the union of all fuzzy ℭ - b- open sets contained in λ.

Proof.

By Definition 5.1, $bInt_{\mathfrak{C}}\,\lambda = \vee\{\mu: \mu \leq \lambda, \mu$ is fuzzy ℭ-b-open$\}$. Taking complement on both sides, we get $\mathfrak{C}\,(bInt_{\mathfrak{C}}(\lambda)(x)) = \mathfrak{C}\,(\sup\{\mu(x): \mu(x) \leq \lambda(x), \mu$ is fuzzy ℭ-b-open$\})$. Since ℭ satisfies the monotonic and involutive conditions, by using Lemma 2.2, $\mathfrak{C}\,(bInt_{\mathfrak{C}}\,(\lambda)(x)) = \inf\{\,\mathfrak{C}(\mu(x)); \mu(x) \leq \lambda(x), \mu$ is fuzzy ℭ-b-open$\}$. By using Definition 2.1, $\mathfrak{C}(bInt_{\mathfrak{C}}\,(\lambda)(x)) = \inf\{\mathfrak{C}\mu(x)): \mathfrak{C}\,\mu(x) \geq \mathfrak{C}\lambda(x), \mu$ is fuzzy ℭ-b-open$\}$. By using Proposition 4.3, ℭμ is fuzzy ℭ-b-closed, by replacing ℭμ by η, we see that $\mathfrak{C}(bInt_{\mathfrak{C}}\,(\lambda)(x)) = \inf\{\eta(x): \eta(x)\geq \mathfrak{C}\,\lambda(x), \mathfrak{C}\eta$ is fuzzy ℭ-b-open$\}$. By using Definition 5.4, $\mathfrak{C}\,(bInt_{\mathfrak{C}}\,(\lambda)(x)) = bCl_{\mathfrak{C}}\,(\mathfrak{C}\,\lambda)\,(x)$. This proves that $\mathfrak{C}(bInt_{\mathfrak{C}}\,\lambda) = bCl_{\mathfrak{C}}\,(\mathfrak{C}\,\lambda)$.

By using Definition 5.4, $bCl_{\mathfrak{C}}\,\lambda = \wedge\{\mu: \lambda\leq\mu, \mu$ is fuzzy ℭ-b-closed$\}$. Taking complement on both sides, we get $\mathfrak{C}\,(bCl_{\mathfrak{C}}\,\lambda\,(x)) = \mathfrak{C}\,(\inf\{\mu(x); \mu(x) \geq \lambda(x): \mu$ is fuzzy ℭ-b-closed$\})$. Since ℭ satisfies the monotonic and involutive conditions, by using Lemma 2.2, $\mathfrak{C}\,(bCl_{\mathfrak{C}}\,\lambda(x)) = \sup\{\,\mathfrak{C}\,(\mu(x)): \mu(x) \geq \lambda\,(x): \mu$ is fuzzy ℭ-b-closed$\}$. By Definition 2.1, $\mathfrak{C}\,(bCl_{\mathfrak{C}}\lambda(x)) = \sup\{\,\mathfrak{C}\mu(x): \mathfrak{C}\,\mu(x)) \leq \mathfrak{C}\,\lambda(x): \mu$ is fuzzy ℭ -b-closed$\}$. By using Proposition 4.3, ℭμ is fuzzy ℭ-b-open, by replacing ℭμ by η, we see that $\mathfrak{C}(bCl_{\mathfrak{C}}\,\lambda(x)) = \sup\{\eta(x): \eta(x) \leq \mathfrak{C}\,\lambda(x); \eta$ is fuzzy ℭ-b-open$\}$. By using Definition 5.1, $(bCl_{\mathfrak{C}}\lambda(x)) = bInt_{\mathfrak{C}}(\mathfrak{C}\lambda)(X)$. This proves $\mathfrak{C}\,(bCl_{\mathfrak{C}}\,(\lambda)) = bInt_{\mathfrak{C}}\,(\mathfrak{C}\,\lambda)$.

Proposition 5.6

Let (X, τ) be a fuzzy topological space and let ℭ be a complement function that satisfies the monotonic and involvtive conditions. Then for the fuzzy subsets λ and μ of a fuzzy topological space X, we have

(i) $\lambda \leq bCl_{\mathfrak{C}}\, \lambda$,
(ii) λ is fuzzy ℭ-b- closed ⇔ $bCl_{\mathfrak{C}}\, \lambda = \lambda$,
(iii) $bCl_{\mathfrak{C}}\, (bCl_{\mathfrak{C}}\lambda) = bCl_{\mathfrak{C}}\, \lambda$,
(iv) If $\lambda \leq \mu$ then $bCl_{\mathfrak{C}}\lambda \leq bCl_{\mathfrak{C}}\, \mu$.

Proof.

The proof for (i) follows from $bCl_{\mathfrak{C}}\, \lambda = \inf\{\mu : \mu \geq \lambda, \mu$ is fuzzy ℭ-b- closed$\}$. Let λ be fuzzy ℭ-b-closed. Since ℭ satisfies the monotonic and involvtive conditions. Then by using Proposition 4.3, ℭλ is fuzzy ℭ-b-open. By using Proposition 5.2, $bInt_{\mathfrak{C}}\, (\mathfrak{C}\, \lambda) = \mathfrak{C}\, \lambda$. By using Proposition 5.5, we see that $\mathfrak{C}\, (bCl_{\mathfrak{C}}\lambda) = \mathfrak{C}\lambda$. Taking complement on both sides, we get $\mathfrak{C}(\mathfrak{C}(bCl_{\mathfrak{C}}\lambda)) = \mathfrak{C}\, (\mathfrak{C}\, \lambda)$. Since the complement function ℭ satisfies the involutive condition, $bCl_{\mathfrak{C}}\lambda = \lambda$.

Conversely, we assume that $bCl_{\mathfrak{C}}\, \lambda = \lambda$. Taking complement on both sides, we get $\mathfrak{C}(bCl_{\mathfrak{C}}\, \lambda)) = \mathfrak{C}\lambda$. By using Proposition 5.5, $bInt_{\mathfrak{C}}\mathfrak{C}\lambda = \mathfrak{C}\lambda$. By using Proposition 5.2, ℭλ is fuzzy ℭ -b-open. Again by using Proposition 4.3, λ is fuzzy ℭ-b-closed. Thus (ii) proved.

By using Proposition 5.5, $\mathfrak{C}(bCl_{\mathfrak{C}}\lambda) = bInt_{\mathfrak{C}}\, (\mathfrak{C}\lambda)$. This implies that $\mathfrak{C}\, (bCl_{\mathfrak{C}}\, \lambda)$ is fuzzy ℭ-b-open. By using Proposition 4.3, $bCl_{\mathfrak{C}}(\lambda)$ is fuzzy ℭ-b- closed. By applying (ii), we have $bCl_{\mathfrak{C}}\, (bCl_{\mathfrak{C}}\, \lambda) = bCl_{\mathfrak{C}}\, \lambda$. This proves (iii).

Suppose $\lambda \leq \mu$. Since ℭ satisfies the monotonic condition $\mathfrak{C}\lambda \geq \mathfrak{C}\mu$. This implies that $bInt_{\mathfrak{C}}\mathfrak{C}\lambda \geq bInt_{\mathfrak{C}}\mathfrak{C}\mu$.Taking complement on both sides, we get $\mathfrak{C}\, (bInt_{\mathfrak{C}}\mathfrak{C}\lambda) \leq \mathfrak{C}(bInt_{\mathfrak{C}}\mathfrak{C}\mu)$. Then by using Proposition 5.5, $bCl_{\mathfrak{C}}\, \lambda \leq bCl_{\mathfrak{C}}\, \mu$. This proves (iv).

Proposition 5.7

Let (X, τ) be a fuzzy topological space and let ℭ be a complement function that satisfies the monotonic and involutive conditions. Then for any two fuzzy subsets λ and μ of a fuzzy topological space, we have (i) $bCl_{\mathfrak{C}}\, (\lambda \vee \mu) = bCl_{\mathfrak{C}}\, \lambda \vee bCl_{\mathfrak{C}}\, \mu$ and (ii) $bCl_{\mathfrak{C}}\, (\lambda \wedge \mu) \leq bCl_{\mathfrak{C}}\, \lambda \wedge bCl_{\mathfrak{C}}\mu$.

Proof.

Since ℭ satisfies the involutive condition, $bCl_{\mathfrak{C}}\, (\lambda \vee \mu) = bCl_{\mathfrak{C}}\, (\mathfrak{C}\, (\mathfrak{C}\, (\lambda \vee \mu)))$. Since ℭ satisfies the monotonic and involutive conditions, by using Proposition 5.5, $bCl_{\mathfrak{C}}\, (\lambda \vee \mu) = \mathfrak{C}\, (bInt_{\mathfrak{C}}\, (\mathfrak{C}\, (\lambda \vee \mu)))$. By using Lemma 2.2, we have $bCl_{\mathfrak{C}}\, (\lambda \vee \mu) = \mathfrak{C}(bInt_{\mathfrak{C}}\, (\mathfrak{C}\lambda \wedge \mathfrak{C}\mu))$. Again by using Lemma 2.2, $Cl_{\mathfrak{C}}(\lambda\vee\mu) \leq \mathfrak{C}((bInt_{\mathfrak{C}}\mathfrak{C}\lambda)\wedge(bInt_{\mathfrak{C}}\mathfrak{C}\mu)) = \mathfrak{C}(bInt_{\mathfrak{C}}\mathfrak{C}\lambda)\vee\mathfrak{C}(bInt_{\mathfrak{C}}\mathfrak{C}\mu)$. By using Proposition 5.5, $bCl_{\mathfrak{C}}(\lambda\vee\mu) \leq bCl_{\mathfrak{C}}\lambda\vee bCl_{\mathfrak{C}}\mu$. Also $bCl_{\mathfrak{C}}(\lambda) \leq bCl_{\mathfrak{C}}(\lambda\vee\mu)$ and $bCl_{\mathfrak{C}}(\mu) \leq bCl_{\mathfrak{C}}(\lambda\vee\mu)$ that implies $bCl_{\mathfrak{C}}(\lambda)\vee bCl_{\mathfrak{C}}(\mu) \leq bCl_{\mathfrak{C}}(\lambda\vee\mu)$. Then it follows that $bCl_{\mathfrak{C}}(\lambda\vee\mu) = bCl_{C}\lambda\vee bCl_{C}\mu$. Since $bCl_{C}(\lambda\wedge\mu) \leq bCl_{C}\lambda$ and $bCl_{C}(\lambda\wedge\mu) \leq bCl_{C}\mu$, it follows that $bCl_{\mathfrak{C}}(\lambda\wedge\mu) \leq bCl_{\mathfrak{C}}\lambda\wedge bCl_{\mathfrak{C}}\mu$.

Proposition 5.8

Let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then for any family $\{\lambda_\alpha\}$ of fuzzy subsets of a fuzzy topological space, we have (i) $\vee(bCl_{\mathfrak{C}}\lambda_\alpha) \leq bCl_{\mathfrak{C}}(\vee\lambda_\alpha)$ and (ii) $bCl_{\mathfrak{C}}(\wedge\lambda_\alpha) \leq \wedge(bCl_{\mathfrak{C}}\lambda_\alpha)$

Proof.

For every β, $\lambda_\beta \leq \vee\lambda_\alpha \leq bCl_{\mathfrak{C}}(\vee\lambda_\alpha)$. By using Proposition 5.6, $bCl_{\mathfrak{C}}\lambda_\beta \leq bCl_{\mathfrak{C}}(\vee\lambda_\alpha)$ for every β. This implies that $\vee bCl_{\mathfrak{C}}\lambda_\beta \leq bCl_{\mathfrak{C}}(\vee\lambda_\alpha)$. This proves (i). Now $\wedge\lambda_\alpha \leq \lambda_\beta$ for every β. Again using Proposition 5.6, we get $bCl_{\mathfrak{C}}(\wedge\lambda_\alpha) \leq bCl_{\mathfrak{C}}\lambda_\beta$. This implies that $bCl_{\mathfrak{C}}(\wedge\lambda_\alpha) \leq \wedge bCl_{\mathfrak{C}}\lambda_\alpha$. This proves (ii).

REFERENCES

[1] K. K. Azad, On fuzzy semi-continuity, Fuzzy Almost continuity and Fuzzy weakly continuity, *J.Math.Anal.Appl.* 82, 14-32(1981).

[2] K.Bageerathi, G.Sutha, P.Thangavelu, A generalization of fuzzy closed sets, *International Journal of fuzzy systems and rough systems*, 4(1)(2011), 1-5.

[3] K.Bageerathi, P.Thangavelu, A generalization of fuzzy Boundary, *International Journal of Mathematical Archive* – 1(3)(2010), 73-80.

[4] K.Bageerathi, P.Thangavelu, On nearly fuzzy **C**-closed sets in fuzzy topological spaces, Theoretical Mathematics and Applications, (submitted)

[5] S.S.Benchalli and Jenifer Karnel, On Fuzzy b-open sets in Fuzzy Topological Space**s** *J.Computer and Mathematical Sciences* 1, 127-134 (2010).

[6] A. S. Bin Shahana, On fuzzy strong semi continuity and fuzzy precontinuity, *Fuzzy sets and systems* 44, 303-308 (1991).

[7] C.L.Chang, Fuzzy topological space, *J.Math.Anal.Appl.*24 (1968), 182-190.

[8] George J.Klir and Bo Yuan, Fuzzy Sets and Fuzzy Logic Theory and Applications, *Prentice – Hall, Inc,* 2005.

[9] Pao-Ming Pu and Ying- Ming Liu, Fuzzy Topology- I.Neighborhood structure of fuzzy point and Moore-smith convergence, *J. Math.Anal.Appl.* 6**,** 571-599 (1980).

[10] S.S.Thakur and Surendra Singh, On fuzzy semi-preopen sets and fuzzy semi-Precontinuity, *Fuzzy sets and systems* 98**,** 383-391(1998).

[11] C.K.Wong, Fuzzy topology: Product and quotient theorems, 5, 512-521 (1974).

[12] L. A. Zadeh, Fuzzy sets, *Inform. Contr* . 8, 338-353 (1965).