Extension of g -Fuzzy Product Topology- 1

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Abstract:

The paper is a continuous discussion about g- fuzzy product topological spaces introduced in our earlier paper [1]. In this paper, the purpose is to give an extension of g-fuzzy Product topology.

Key words:

Fuzzy set, fuzzy point, g- fuzzy topology, Base for g-fuzzy topological space, g-fuzzy Subspace Topology, g- fuzzy compactness, g- fuzzy Hausdorff space, product spaces, g- fuzzy box topology, g- fuzzy product topology.

1. Introduction

Mathews and Samuel [9], introduced an alternate and more general definition of fuzzy topological spaces called g-fuzzy topological spaces using the two additive operations sum $\oplus$ and conjunction &. Here we give an extension of Product topology in g- fuzzy topological spaces.

2. Basic Concepts

In this section we give several definitions and some of their consequences relevant to this paper

Definition 2.1 A fuzzy set A in X is characterized by a membership function $\mu : X \rightarrow [0,1]$ where [0,1] is the closed unit interval, while an ordinary set $A \subseteq X$ is identified with its characteristic function $\chi_A: X \rightarrow \{0, 1\}$. If A is a fuzzy set in X, its membership function is dented by $\mu_A(x)$.

Let I(X) be the family of all the fuzzy sets in X called fuzzy space and P(X) be the class of fuzzy sets whose membership functions have all their values in {0,1}.

Definition 2.2 A *fuzzy point* λ in a set X is a fuzzy set in X given by

$$\mu_\lambda(x) = t \text{ for } x = x_\lambda \quad (0 < t < 1)$$

and

$$\mu_\lambda(x) = 0 \text{ for } x \neq x_\lambda .$$

The point x_λ is called the *support* of λ and t the value of λ. The fuzzy point x_λ is said to belong to a fuzzy set A, denoted by $x_\lambda \in A$ iff $\lambda \leq A(x)$ for all $x \in X$.

Definition 2.3

i. The sum of two fuzzy sets A and B in a set X ,denoted by $A \oplus B$,is a fuzzy set in X defined by $(A \oplus B)(x) = \min(1, A(x) + B(x))$ for all

ii. The conjunction of two fuzzy sets A and B ,denoted by A&B , is a fuzzy set in X defined by $(A \& B)(x) = \max(0, A(x) + B(x) - 1)$ for all $x \in X$

Now we give the definition of the sum $\oplus$ and conjunction & for an indexed family of fuzzy sets as follows:

Definition 2.4

Let J be an infinite index set and $J_i \subset J$ be finite or countable set.

Similar to operations on ordinary sets, we can generalize the *sum* of any family $\{A_i \ / \ i \in J\}$ of fuzzy sets of a non empty set X as $(\oplus_{i \in J} A_i)(x) = \sup_{J_i \subset J}((\oplus_{i \in J_i} A_i)(x)$ for $x \in X$

In a similar way we define the *conjunction* of any family $\{\mu_i \ / \ I \in J\}$ of fuzzy sets of a non empty set X as

$$(\&_{i \in J} A_i)(x) = \inf_{J_i \subset J}((\&_{i \in J_i} A_i)(x) \text{ for } x \in X$$

Definition 2.5 Let f be a function from X to Y. Let B be a fuzzy set in Y with membership function μ_B. Then the *inverse* of B, written as $f^{-1}(B)$, is a fuzzy set in X whose membership function is defined by $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ for all x in X.

3. g- Fuzzy Topological Spaces

Definition 3.1 A *g-fuzzy topology* on X is a collection T if

1 $\Phi, X \in T$
2 $A, B \in T$ implies $A \& B \in T$ and
3 For any subfamily $\{A_\alpha\}_{\alpha \in J}$ in T implies $(\oplus_\alpha A_\alpha) \in T$

Members of T are called g- *fuzzy open sets* and the pair (X, T) is called a g-*fuzzy topological space* or gfts in short. Complements of g- fuzzy open sets are called g- *fuzzy closed sets*.

Remark 3.2 If A and B are ordinary sets in X, then $A \oplus B = A \cup B$, $A \& B = A \cap B$ and $A \Theta B = A \setminus B$. Thus the ordinary topology becomes special case of g- fuzzy topology.

Definition 3.3 Let (X,T) and (Y,S) be two g-fuzzy topological spaces and $f : X \to Y$ be a function. Then f is said to be a g-fuzzy continuous function if $f^{-1}(V) \in T$ for each $V \in S$.

Definition 3.4 Let (X, T) be a gfts. A sub family ß of T is called a *base* for T, if and only if, for each A in T, there exists $(A_i)_{i\in J} \subset$ ß such that $A = (\oplus_{i\in J} A_i$

Definition 3.5 Let A and $\{A_\alpha\}_{\alpha\in J}$ be fuzzy sets in X .Then $\{A_\alpha : \alpha\in J\}$ is called a cover of A iff $\oplus\{A_\alpha : \alpha\in J\} \supset A$. If there exists a subset J_1 of J such that $\oplus \{A_\alpha : \alpha\in J_1\} \supset A$, then $\{A_\alpha : \alpha\in J\}$ is called a sub cover.

Definitions 3.6 Let (X, T) be a g-fts and $Y \subset X$; then we call the family
$T_Y = \{U \,\&\, Y: U \in T\}$ which is a g-fuzzy topology for Y, the relative g- fuzzy topology.

T_Y contains Φ and Y because Φ = Y & Φ and Y = Y & X, Φ and X are elements of T.
It is closed under finite disjunctions and arbitrary sums of g-fuzzy open sets follows from the equations
$\&_{i=1}^{n} (A_i \,\&\, Y) = (\&_{i=1}^{n} A_{i)} \,\&\, Y$ and $\oplus_{i\in J} (A_i\& Y) = (\oplus_{i\in J} A_i) \,\&\, Y$

Such a fuzzy topological space (Y, T_Y) is called a *g- fuzzy subspace* of (X, T).

Theorem 3.7 If ß is a base for the f-fuzzy topology of X, then the collection
$ß_Y = \{B\& Y / B \in ß\}$ is a base for the g-fuzzy subspace topology on Y.

Proof: Let U be a g-open fuzzy set in X and $y \in U \,\&\, Y$. We can choose an element B of ß such that $y\in B \subset U$. Then, $y \in B \,\&\, Y \subset U \,\&\, Y$. Hence $ß_Y$ is a base for the g-fuzzy subspace topology on Y.

Theorem 3.8 Let Y be a g-fuzzy subspace of X. If U is g- fuzzy open in Y and Y is g- open in X, and then U is g- open in X

Proof: Since U is g- fuzzy open in Y, U = Y & V for some fuzzy set V g- open in X. since both Y and V both g-fuzzy open in X, so is Y&V.

Definition 3.9 A gfts (X,δ) is said to have the Hausdorff property or to be a Hausdorff if for each pair x , y ∈ X with x ≠ y , implies that there exist fuzzy open sets μ and ν with $\mu(x) = \underline{1} = \nu(y)$ and $\mu \,\&\, \nu = \underline{0}$.

4. G-fuzzy product topology on X × Y

In this section, we define a g-fuzzy topology on the product X x Y of two g- fuzzy topological spaces. Then we generalize this definition to arbitrary Cartesian products.

Definition 4.1 Let A and B be fuzzy sets in X and Y, respectively. The *cartesian product* A × B of A and B is a fuzzy set in X × Y defined by

$$(A\times B)(x, y) = \min\{A(x), B(y)\} \text{ for every } (x, y) \in X \times Y$$

Example 4.2 Let X = Y = I. Consider fuzzy sets A and B of I defined as

$A(x) = 1/6$, if $x = 2/3$ and 0 otherwise.
$B(x) = 2/5$, if $x = 4/5$ and 0 otherwise.

Then, $A\times B$ is given by $(A\times B)(x, y) = \min\{1/6, 2/5\} = 1/6$ if $(x, y) = (2/3, 4/5)$ and 0 otherwise.

Theorem 4.3 [1] Let f be a mapping from a set X to a set Y and let A and B be two fuzzy sets in Y then

1. $f^{-1}(A \oplus B) = f^{-1}(A) \oplus f^{-1}(B)$
2. $f^{-1}(A \,\&\, B) = f^{-1}(A) \,\&\, f^{-1}(B)$

This result can be extended to a family of fuzzy sets

Theorem 4.4 Let f be a mapping from a set X to a set Y and let $\{B_j\}_{j\in J}$ be a family of fuzzy sets in Y then

1. $f^{-1}(\oplus_{j\in J} B_j) = \oplus_{j\in J} f^{-1}(B_j)$
2. $f^{-1}(\&_{j\in J} B_j) = \&_{j\in J} f^{-1}(B_j)$

Proof: Follows from the definition

Definition 4.5 Let X and Y be g- fuzzy topological spaces. The g- *fuzzy product topology on* $X \times Y$ is the topology having as basis the collection β of all fuzzy sets of the form $U \times V$ where U is g-fuzzy open set of X and V is a g-fuzzy open set of Y.

The collection β is a basis because $X \times Y$ is itself a basis element and the intersection of any two basis elements $U \times V$ and $U_1 \times V_1$ is another basis element: For

$$(U \times V) \,\&\, (U_1 \times V_1) = (U \,\&\, U_1) \times (V \,\&\, V_1)$$

is a basis element because $U \,\&\, U_1$ and $(V \,\&\, V_1)$ are g-fuzzy open in X and Y respectively.

Theorem 4.6 If A is a g- fuzzy subspace of X and B is a g-fuzzy subspace of Y, then the g-fuzzy product topology on $A \times B$ is the same as the g- fuzzy topology $A \times B$ inherits as a fuzzy subspace of $X \times Y$

Proof: The fuzzy set $U \times V$ is the general basis element for $X \times Y$ where U is g-fuzzy open set of X and V is a g-fuzzy open set of Y. Therefore $(U \times V) \,\&\, (A \times B)$ is the general basis element for the g- fuzzy subspace topology on $A \times B$. Now

$$(U \times V) \,\&\, (A \times B) = (U \,\&\, A) \times (V \,\&\, B).$$

Since $U \,\&\, A$ and $V \,\&\, B$ are the general open sets for the subspace topologies on A and B respectively, the set $(U \,\&\, A) \times (V \,\&\, B)$ is the general basis element for the g- fuzzy product topology on $A \times B$.
Thus, the bases for the g- fuzzy subspace topology on $A \times B$ and for the g- fuzzy product topology on $A \times B$ are the same. Hence the topologies are the same.

Theorem 4.7 If X and Y are g fuzzy Hausdorff spaces and Y is g- fuzzy compact, then the projection mapping π_1: X x Y $\rightarrow$ X maps g- fuzzy closed sets onto g- fuzzy closed sets.

Definition 4.8 Let $\{X_\alpha : \alpha \in J\}$ be a family of spaces. Let $X = \Pi_{\alpha \in J} X_\alpha$ be the usual *product space* and let π_α be the *projection* from X onto X_α.

The definition of g-fuzzy product topology on X x Y may be extended to arbitrary product of a family of spaces. There are two ways of generalizing the definition.

One way to impose a g-fuzzy topology on the product space is the following; it is a direct generalization of the way we defined a basis for the product topology on X x Y.

Definition 4.9 Let $\{X_\alpha : \alpha \in J\}$ be a family of g- fuzzy topological spaces. Let us take as a basis for a fuzzy topology on the product space $\Pi_{\alpha \in J} X_\alpha$ the collection of all fuzzy sets of the form $\Pi_{\alpha \in J} U_\alpha$, where U_α is g- fuzzy open in each X_α: $\alpha \in J$. The g-fuzzy topology generated by this basis is called the g- *fuzzy box topology*.

Second way to impose a fuzzy topology on a product space is given in the following definition.

Definition 4.10 Let $\{(X_\alpha, T_i) : \alpha \in J\}$ be a family of g-fts and let $X = \Pi_\alpha X_\alpha$ be the Cartesian product of X_α's. Let projection$\pi_{\alpha:}$ X$\rightarrow$ X_α be the projection function. Then the g-*fuzzy product topology* $\Pi_\alpha T_\alpha$ on X is the smallest g- fuzzy topology on X (if it exists) which makes each projection π_α , g- fuzzy continuous.

Theorem 4.11 The g- *fuzzy box topology* on the product space $\Pi_{\alpha \in J} X_\alpha$ has a basis all fuzzy sets of the form $\Pi_{\alpha \in J} U_\alpha$, where U_α is g- fuzzy open in each X_α: $\alpha \in J$. The g-*fuzzy product topology* on the product space $\Pi_{\alpha \in J} X_\alpha$ has as basis all fuzzy sets of the form $\Pi_{\alpha \in J} U_\alpha$, where U_α is g- fuzzy open in X_α for each $\alpha \in J$ and $U_\alpha = X_\alpha$ except for finitely many values of $\alpha \in J$.

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